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On the solution of games with arbitrary payoffs: An application to an over-the-counter financial market

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Abstract

This paper defines a variety of game theoretic solution concepts in the language of soft set theory. We begin by defining the Nash equilibrium in pure strategies. We assume that the gains of the players are totally ordered and non-desirable alternatives are absent. Moreover, we introduce the notions of strong and semi-strong utility. These two completely new notions, serve as a mechanism for converting non-ordered gains into totally ordered ones. We define the Nash equilibrium in mixed strategies in a general framework by introducing the notion of an extended game and strategy space. We finally define the Nash solution to cooperative bargaining games within the framework of soft set theory, illustrate a practical application to an over-the-counter (OTC) financial market, and provide a detailed numerical example.

Keywords: Game theory; Soft set theory; Nash equilibrium; Cooperative bargaining games; Over-the-counter financial markets; Financial intermediation

JEL Classification: C6, C7, G1

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1. Introduction

It has been shown that fuzzy sets (Zadeh, 1965), probability theory, rough sets (Pawlak, 1982) and other mathematical tools have inherent difficulties in dealing with uncertainties and modeling complexities in economics, engineering and other scientific fields. As Molodtsov (1999) points out, the main reason for the existence of these difficulties is the inadequacy of the parametrization tool of those theories. In light of these mathematical rigidities Molodtsov (1999) proposed a new novel approach, so-called soft set theory, for modeling vagueness and uncertainty. Soft sets can be regarded as a special case of context-dependent fuzzy sets, as defined in Thielle (1999). Soft set theory can be applied to different fields of research, ranging from operations research and measurement theory to Riemann integration and game theory, so in that sense it is an interdisciplinary mathematical approach.

In classical mathematics, in order to deal with problems under uncertainty, we construct models of an object and define the exact solution of these mathematical models. Usually this solution is associated with a high degree of complexity so that the idea of an approximate solution is introduced. On the other hand, in soft set theory, the initial description of an object has an approximate nature which is free from any restrictions, thus making this theory particularly appealing and easily applicable in practice. Any parametrization we prefer using words and sentences, real numbers, and functions, can be used to describe the object.

Since the seminal work of Molodtsov (1999), research on soft set theory has been progressing at a rapid pace. Maji et al. (2002, 2003) present an application of soft sets in decision making problems using the rough mathematics proposed by Pawlak (1982) and prove various theoretical propositions on soft set operations. Chen et al. (2005) propose a definition of parameterization reduction of soft sets and compare it with the concept of attributes reduction in rough sets theory. Aktaş and Çağman (2007) compare soft sets to fuzzy and rough sets, provide a definition of soft groups, and derive their basic properties following Molodtsov's definition of soft sets. Çağman and Enginoğlu (2010) use newly defined products of soft sets and *uni-int* decision functions to construct *uni-int* decision making methods applicable to problems with embedded uncertainties [4].

[4] Other studies on soft set theory applications include those of Bakshi et al., 2016; Murthy and Maheswari, 2017; Kandemir, 2018; Aygün and Kamacı, 2019; Dalkılıç, 2021; Akram et al., 2021; Maharana and Mohanty, 2021.

Recent reviews of academic papers written on soft set theory applications include those of Zahedi Khameneh and Kılıçman (2019) and Zhan and Alcantud (2019). Zahedi Khameneh and Kılıçman (2019) provide a survey study of multi-attribute decision making based on soft set theory. Multi-attribute decision making is one class of multiple criteria decision making problems in which the domain of alternatives is discrete, and requires to attribute comparisons involving some type of trade-off between pre-specified alternatives, either implicitly or explicitly. Conventional multiple criteria decision making techniques focus on group decisions as opposed to strategic games (conflicts) in which each decision maker makes individual choices that together determine the outcome. Zhan and Alcantud (2019) provide a survey of parameter reduction algorithms for soft sets. The objective of these algorithms is to reduce the cardinality or size of the parameter set. The scope of this reduction lies primarily in the implied reduction of computationally costly tests that are necessary to determine a solution to a decision making process. The primary difference between these algorithms and our work lies in the fact that in a game, expressed here as a soft set, the parameter set coincides with the set of all strategy combinations or strategy profiles and this set is irreducible. In other words, one cannot dispense with a strategy combination but instead has to search through each combination in order to determine a game's solution. This is true regardless of whether a game is represented as a soft set or not.

Apart from works based on classical soft set theory, there are various hybrid approaches that combine elements from soft set models and other mathematical models. For example, Maji et al. (2001) introduce the concept of fuzzy soft sets by combining soft sets with fuzzy sets. Majumdar and Samanta (2010) propose the generalized fuzzy soft sets, whilst Feng et al. (2010) suggest three different types of hybrid models, namely rough soft sets, soft rough sets, and soft-rough fuzzy sets. Surprisingly, research on game theory using soft set applications is very limited. Deli and Çağman (2016a) introduce the definitions of dominated strategy, saddle point solution, and Nash equilibrium (NE) in pure strategies for a two-player game giving a natural extension to *n*-person games. Deli and Çağman (2016b) give a probabilistic equilibrium solution method of two-person soft games (tps-games). Fernández et al. (2018) propose an application of soft sets to describe coalitions in cooperative games which permits to study new situations of asymmetric players in games.[5]

[5]There are a number of studies that have dealt with zero-sum games with fuzzy payoffs and fuzzy goals game theory, such as those of Maeda (2003), Bector et al. (2004), Vijay et al. (2005), and Cevikel and Ahlatçioğlu (2010), among others. Other studies that use

Soft set theory, as a generic mathematical tool, is well-suited to address financial decision-making problems under uncertainty. Although soft set theory is novel, there are a few studies that have used soft sets in financial applications. Xu et al. (2014) use soft set theory to design a parameter reduction method that could complement conventional principal component analysis, aiming at selecting financial ratios for business failure prediction. The proposed method combines elements from statistical logistic regression and soft set decision theory and is applied to real data sets from Chinese listed firms. Xu and Xiao (2016) propose a new forecasting method to predict business failures that is based on soft set theory. The authors introduce a new weighted scheme based on the receiver operating characteristic (ROC) curve theory to obtain suitable weight coefficients for their model. They conclude that their method demonstrates superior performance and higher stability than other competing methods. Alcantud et al. (2017) propose a new method for valuing real estate in Spain which is based on fuzzy soft sets. Their proposed method allows them to assess a variety of assets where data is heterogeneous and at the same time overcome difficulties inherent in more traditional techniques, such as linear multiple regression.

Harode et al. (2018) construct an optimal investment portfolio that exhibits minimum risk and maximum returns using fuzzy soft set theory. Xu et al. (2019) propose a hybrid data mining model of generalized fuzzy soft sets theory-based ensemble credit scoring model. The authors provide evidence that their proposed model can increase computational efficiency without sacrificing classification accuracy. Jacob John (2021) uses the soft set approach of Kharal (2010) to detect whether two companies with different financial characteristics suffer from a serious liquidity problem. The first company exhibits rising profit-earnings ratios during the last fiscal year and a low amount of paid-up capital, whilst the second company exhibits a very volatile share price and a low profit-earning ratio. The authors have coded the company profiles by observing their financial indicators into a soft set using appropriate linguistic labels. As a rule of thumb, if a company's soft set is similar to a standard liquidity problem profile, it can be deduced that the company suffers from liquidity squeezes. Balcı et al. (2022) introduce network-induced soft sets to study the dynamics of a financial stock market with several orders of interaction. To achieve the model's intelligent parameterization, the authors rely on the bilateral connections between economic agents in a financial network, instead

hybrid approaches include those of Açıkgöz and Tas (2016), Chang et al. (2016), Prasertpong (2021), Ali et al. (2022).

of using any other single feature of the network itself.

Our work is advantageous compared to previous works. First, we define for the first time the notions of *strong* and *semi-strong utility*. These notions explore the connection between game theory expressed in a language of soft set theory and "classical" game theory. Second, we present a wide array of game theoretic solution concepts using soft set theory, extending from normal form games to Nash solutions to bargaining problems. Third, the methods we develop are straightforward and unambiguous. As a result, one can always draw an analogy between soft set theory and classical game theory.

This study contributes to the existing literature in a number of ways. First, we define the concept of Nash Equilibrium (NE) within the framework of soft set theory in a way that is consistent with classical game theory, pointing out some of the weaknesses in the definition provided by Deli and Çağman (2016a). To this end, two alternative approaches are considered. In the first approach, we introduce homogeneous player gains that are obtained via total ordering and rejection of "bads", i.e. alternatives that enter a player's gain tuple and result in the reduction of the gains associated with the strategy chosen. In the second approach, we introduce the concepts of strong and semi-strong utility, along with a utility correspondence whose image is the set of all non-negative reals. This novel notion serves as a device that converts non-ordered gains into totally ordered ones and subsequently computes the game's NE.

Our second contribution relates to the translation of the Nash equilibrium in mixed strategies (NEMS) concept into the framework of soft set theory. To achieve this, we depart from a very general setup that extends the players' strategy spaces and the game itself which we now call the extended game. The players' extended strategy spaces are supersets of the original strategy spaces that may contain mixed strategies or other alternative strategies in addition to the already existing ones. Thus the initial pure strategies game is now represented by a map from the Cartesian product of the players' extended strategy spaces to an extension of the game's pure strategies power set. For the extended game we also define the players' best response correspondences. This work is significant because the extended game is compatible with a wide range of fixed point theorems, depending on the structure of the extended players' strategy spaces and the extended game itself. We provide an example that clarifies this abstract setup. We also provide a less abstract example for the computation of a game's NEMS in the language of soft set theory by defining the players' expected pay-off functions as vector functions and associate each gain with a standard basis vector of the Euclidean

4-space.

Finally, we propose an application of soft set theory to cooperative bargaining games in financial markets. Generally speaking, although our theoretical framework is used to describe how a game is played, and in particular, describe and explain a number of game theoretic solution concepts and the predictions made with regard to these concepts, it may also be extremely useful in tackling practical problems in financial markets. We offer a solution to practical bargaining problems in over-the-counter (OTC) financial markets in which the price setting mechanism is not revealed to all market participants and the bargaining process is fundamentally influenced by information asymmetries. Given the complexity of the intermediation process in OTC markets that raises financial stability concerns, our theoretical approach enables us to model the bargaining process between buyers and sellers offering solutions to market liquidity improvements. To this end, we discuss a detailed numerical application in Section 6. We use a market setting similar to that of Duffie et al. (2005) on intermediation in OTC markets[6]. From a practical point of view, the primary advantage of our approach rests in the simplicity of the overall formalism. In particular, in Duffie et al. (2005) the bid, ask, bid-ask spread, and the asset's price are being determined by solving a continuous-time, stochastic optimal control problem. This is only done for the steady state, i.e., for constant masses of all four investor types. Hence, the analysis carried out revolves around stationary equilibria. In the present setup, this is trivially true as the optimization problem in Equation (40) is static from the outset. This simplification is the result of the soft set theoretical approach we take in this study. Given the interdisciplinary nature of game theory, our theoretical framework may be applicable to other practical problems in international finance where bargaining takes

[6] Over-the-counter (OTC) financial markets are less formal and standardised than exchange-traded markets. In OTC markets there are no designated market makers, and dealers usually act informally as market makers by quoting bid and ask prices to other dealers and to their customers. Regardless of the type of negotiation (customer-to-dealer or dealer-to-dealer), bilateral trading takes place in OTC markets and the price setting mechanism is not revealed to all participants (a discussion is provided by Dodd, 2017). OTC markets are also less transparent than exchange-traded markets and subject to fewer regulations. This can negatively affect liquidity in those markets, especially during periods of market stress such as those of the 2007/08 U.S. subprime mortgage crisis and the 2009/12 euro-area sovereign debt crisis, as liquidity dry-ups may force dealers to exit the market with severe consequences for trading, price discovery (Hasbrouck, 1995; Werner and Kleidon, 1996; Baillie et al., 2002; Eun and Sabherwal, 2003; Brandt and Kavajecz, 2004; Mizrach and Neely, 2008), and contagion risk (Allen and Gale, 2000; Forbes and Rigobon, 2002; Vayanos, 2004; Baur and Lucey, 2009; Beber et al., 2009; Longstaff, 2010; Papavassiliou, 2014; Claeys and Vašíček, 2014; De Santis, 2014; Blatt et al., 2015; O'Sullivan and Papavassiliou, 2019). Unlike in organised exchanges, there are no central clearing and settlement mechanisms in place for OTC markets and thus transactions are not guaranteed by an exchange, leading to substantial counterparty risk. In recent years, there has been a substantial increase in OTC derivatives transactions that was made possible due to advances in computer technology. A recent report by the Bank for International Settlements (BIS) reveals that the gross market value of all contracts in global OTC derivates markets in the second half of 2021 reached 12.44 billions of U.S. dollars, while the notional amounts outstanding for the same period reached almost 610 billions of U.S. dollars (Source: BIS Statistics Explorer (http://stats.bis.org/statx/).

place, such as mergers and acquisitions under financial constraints and corporate negotiations that involve contingent payments or securities (Chaves and Varas, 2021).

Duffie et al. (2005) propose a dynamic asset-pricing model that captures features of the bargaining process that takes place in OTC decentralised markets. The authors derive the equilibrium allocations and prices negotiated between investors and demonstrate how these equilibrium relationships depend on investors' bargaining powers and search abilities. Although OTC markets are characterized by the absence of an organized exchange, there can exist market intermediaries that facilitate trading of financial assets. In that sense, an OTC market is not necessarily a frictionless market, as there are transaction costs in the form of bid-ask spreads that must be borne by investors.

Cooperative bargaining games are very relevant to price building in OTC financial markets. As Duffie et al. (2005) explain, when counterparties meet, their relationship is inherently strategic and prices are determined through a bargaining process that reflects each investor's or marketmaker's alternatives to trade. In financial OTC markets, investors bargain over the prices at which they are willing to buy or sell assets. If an asset owner has private information about the asset's quality and liquidity condition, they may be incentivised to hide their motives to sell in an attempt to get a more favorable price. Thus, the bargained price may be influenced by such incentives due to the existence of information asymmetries (Kim, 2019). In such negotiations there can be significant bargaining delays that extend from a few minutes, for very liquid securities such as short-term bonds, to months, as in the real estate market (Tsoy, 2016). In recent studies it has become apparent that the intermediation process in OTC markets is not so straightforward as dealers are heterogeneous with respect to their typical positions, the frequency with which they trade, and the prices at which they transact, which further affects the price building process in those markets (Hugonnier et al., 2020).

We offer an elegant solution to the aforementioned bargaining game using soft set theory. Solving such type of game in this framework implies choosing elements in the players' upper sets, such that their union is equal to the game's universe set. The equilibrium solution of the game need not be unique. If more than one solution exist, then choosing the optimal solution requires the existence of an ordering relation. Our novel modeling approach contributes to the cooperative bargaining literature as it offers a different perspective on intermediation in financial markets, such as foreign exchange, bond, equity, and mortgage-backed securities markets. We explicitly model transaction costs in the form of bid-ask spreads that are present in these markets and demonstrate how they become narrower, i.e. market liquidity is enhanced, when investors can find each other more easily. Our findings can have important implications for regulators and policymakers who design and implement reforms in trading systems with the aim to improve market liquidity. Given that game theory is used in many fields such as economics, finance, political science, and psychology, the findings of this research can be used as an interdisciplinary benchmark that stretches beyond the boundaries of a single research field.

One might ask why soft sets are useful to economic and game theorists. Why should one invest time and effort in a new tool or "technology"? The answer to this question is that soft set theory offers solutions to games in which the assumptions of "classical" game theory, such as compactness, convexity, and continuity are not fulfilled. For example, this is the case in discrete games where functions are defined on grids instead of continuous sets. This leads to useful directions in the development of a general theory and its applications.

The rest of the paper is organized as follows. Section 2 presents the basic definitions of soft set theory. Section 3 introduces the concept of Nash equilibrium with homogeneous gains obtained by total ordering and rejection of non-desirable alternatives. Section 4 introduces the concepts of strong utility, semi-strong utility and an associated utility correspondence. Section 5 discusses Nash equilibrium in mixed strategies. Section 6 discusses the Nash bargaining solution using soft set theory, presents an application of soft set theory to cooperative bargaining games, and describes a detailed numerical example from an OTC market. Section 7 concludes the paper.

2. Preliminaries

In this section, we present the basic definitions of soft set theory that are useful for subsequent discussions. Let U be an initial universe set and let E be a set of parameters. According to Molodtsov (1999), a soft set is defined as follows:

DEFINITION 1. A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U, i.e. $F : E \rightarrow P(U)$, where $P(U)$ is the power set of U.

That is, the soft set is a parametrized family of subsets of the set U. Every set $F(\varepsilon)$, $\varepsilon \in E$, may be considered as the set of ε-elements of the soft set (F, E), or as the set of ε-approximate elements of the soft set. Stated differently, we can consider a universe set X, and the power set of X denoted by 2^X that define a map $F : A \to 2^X$, where A is the set of parameters defined in the problem at hand. Then we call the pair (F, A) a soft set over X. Using this definition, a soft set can effectively describe a complex financial problem by defining a universe set of possible outcomes, for instance $X = \{c_1, c_2, c_3, c_4, c_5\}$, a set of parameters such as $A = \{$*high market risk*; *low market risk*; *high systemic risk*; *low systemic risk*$\}$, and a mapping $F : A \to 2^X$ which maps each element of the set A to some subset of the universe set X. The various sets that form the outputs of F are arbitrarily chosen thus they may be empty or have a non-empty intersection.

Molodtsov (1999) offers an approach to construct a new mathematical tool which can be applied to problems with game theory. If $P \subset S$, then $F_i(P, \varepsilon)$ is a set of ε-optimal situations for player i, where P is a subset of admissible strategies, ε is a parameter, $\varepsilon \in E_i$, and S is a set of situations, $S = S_1 \times ... \times S_n$, with S_i being a set of strategies of player i. This game is a soft game with the following notation:

$$\langle (F_{i,} E_i), S_i,\ i = 1, ..., n \rangle \tag{1}$$

where $(F_{i,}E_i) : M(S) \to S, (F_i, E_i)$, and where $M(S)$ denotes a set of all subsets of the set S. The analogue of the Nash equilibrium takes the following form:

DEFINITION 2. Situation $s \in S$ is called a situation of soft ε-equilibrium, $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $\varepsilon_i \in E_i$, if and only if $s \in F_i(s_1 \times ... \times s_{i-1} \times S_i \times s_{i+1} \times ... \times s_n, \varepsilon_i)$, for every $i = 1, ..., n$. Denoting the set of all situations of soft ε-equilibrium by $N(\varepsilon)$, we can call the soft set

$$(N, E_1 \times ... \times E_n), \tag{2}$$

a soft equilibrium. Although Definition 2 may seem daunting at first glance, it is solely based on the notion of the set $F_i(P, \varepsilon)$, with $P \subset S$, as described earlier. Given a parameter ε, the set $F_i(P, \varepsilon)$ as described above, is the set of all optimal strategies or situations for each player i. Hence, when a strategy s belongs to this set and is true for all players, then this strategy has to be optimal and thus constitutes a Nash equilibrium in this set-up.

3. Nash equilibrium (NE) using soft sets: Homogenization of gains via total ordering and no "bads"

If we consider a two-player game and define $S_i, i \in \mathbb{N}, 1 \le i \le 2$, to be the strategy space of player i, then a normal form game for player $i \in \{1,2\}$ in the framework of soft set theory is a pair (F_i, S), where $S := S_1 \times S_2$ is the Cartesian product of the strategy spaces S_1, S_2 and $F_i : S \to 2^U$ is a map from S to the power set 2^U, where U is the universe set containing all possible alternatives for each player in the game. If, for example, the strategy space of players 1 and 2 consists of two strategies, s_1, s_2 and the universe set of all possible alternatives is $U = \{u_1, u_2, u_3, u_4\}$, then S is the set $S_1 \times S_2 = \{s_1, s_2\} \times \{s_1, s_2\} = \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}$ and $F_i : S \to 2^U$ is a mapping which maps each element in $S_1 \times S_2$ to one of the $2^4 = 16$ elements contained in the power set 2^U.

Contrary to the work in Deli and Çağman (2016a), in this study we make the assumption that player gains are totally ordered. This assumption allows us to treat sets as real numbers for which the completeness property holds. Hence it enables us to decide unambiguously which gain set is preferred by each player individually. Along these lines, we introduce a second assumption later in the paper.

Consider a game with two players named 1 and 2. The strategy space of both player 1 and player 2 is the set $S_1 = S_2 = \{s_1, s_2\}$. The universe set of gain alternatives is $U = \{u_1, u_2, u_3, u_4\}$. Player's 1 soft set is the pair:

$$\left(F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{3}$$

We assume that the map $F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}$ is the following:

$$\left\{F_1(s_1, s_1) = \{u_1\},\ F_1(s_1, s_2) = \{u_1, u_2\},\ F_1(s_2, s_1) = \{u_1, u_2, u_3\},\ F_1(s_2, s_2) = \{u_1, u_2, u_3, u_4\}\right\} \tag{4}$$

Player's 2 soft set is the pair:

$$\left(F_2 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{5}$$

We assume that the map $F_2 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}$ for player 2 is the following:

$$\left\{F_2(s_1,s_1)=\{u_1,u_3\},\ F_2(s_1,s_2)=\{u_1,u_2,u_3\},\ F_2(s_2,s_1)=\{u_1\},\ F_2(s_2,s_2)=\{u_1,u_2,u_3,u_4\}\right\} \tag{6}$$

Player's 1 gains are totally ordered in the length-4 chain

$$\{u_1\}\subseteq\{u_1,u_2\}\subseteq\{u_1,u_2,u_3\}\subseteq\{u_1,u_2,u_3,u_4\} \tag{7}$$

Similarly, player's 2 gains are totally ordered in the length-4 chain

$$\{u_1\}\subseteq\{u_1,u_3\}\subseteq\{u_1,u_2,u_3\}\subseteq\{u_1,u_2,u_3,u_4\} \tag{8}$$

Computing the Nash equilibrium (NE) in this case is quite straightforward. Indeed, if player 1 thinks that player 2 will choose strategy s_1, then the best response for 1 is to choose s_2, since $\{u_1,u_2,u_3\}\supseteq\{u_1\}$. If player 1 thinks player 2 will choose s_2, then her best response is to choose s_2 as well, since $\{u_1,u_2,u_3,u_4\}\supseteq\{u_1,u_2\}$. Hence, player 1 always chooses strategy s_2. Similarly, if player 2 thinks player 1 will choose strategy s_1, then her best response is strategy s_2 since $\{u_1,u_2,u_3\}\supseteq\{u_1,u_3\}$. If player 2 believes player 1 will choose s_2, then player's 2 best response is to choose s_2 as well, since $\{u_1,u_2,u_3,u_4\}\supseteq\{u_1\}$. Hence both players 1 and 2 always choose strategy s_2. Therefore, the combination of strategies $(s_1,s_2)\in S_1\times S_2=\{s_1,s_2\}\times\{s_1,s_2\}=\{(s_1,s_1),(s_1,s_2),(s_2,s_1),(s_2,s_2)\}$ is the NE in pure strategies.

However, an additional assumption must be satisfied. In particular, it must be the case that no elements contained in the universe set are bad, i.e. non-desirable. This assumption is not met in Deli and Çağman (2016a). A bad is any alternative that reduces a player's gain and as such is non-desirable by the player. Formally, an alternative $u_j\in U$ is defined as bad for player $i\in\mathbb{N}$ if and only if, gain $X\subset U$ is preferred over $\left\{u_j\right\}\cup X$.

If the assumption of no bads is not satisfied, then $\{u_1,u_2,u_3\}\supseteq\{u_1,u_3\}$ is not sufficient for a player to choose $\{u_1,u_2,u_3\}$ over $\{u_1,u_3\}$. If the alternative prize u_2 is bad, then $\{u_1,u_2,u_3\}\supseteq\{u_1,u_3\}$ but the player will choose the alternative $\{u_1,u_3\}$ over $\{u_1,u_2,u_3\}$. It follows that lack of total ordering of alternatives leads to ambiguities as arbitrary alternatives must be compared in order to compute the NE of the game. For example, it is not possible to identify which of the two alternatives $\{u_1\}$ and $\{u_2\}$ a player will choose, given the fact that these alternatives are neither ordered nor bad. In the following section we introduce the notions of strong and semi-strong utility as a device that converts initially non-totally ordered alternatives into totally ordered ones. In doing so, computing a game's NE becomes feasible.

4. Strong and semi-strong utility

The goal of this section is to explore the connection between game theory expressed in a language of soft set theory and classical game theory. To this end, we initiate a mapping of the type $2^U \xrightarrow{w} \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} :=\{x \in \mathbb{R} : x \geq 0\}$ is the set of all non-negative real numbers. The power set 2^U is the set of all alternative gains and w is a correspondence that maps gains to non-negative reals. This correspondence reveals how a particular player perceives each gain tuple, i.e. each subset of the universe set, U. Such correspondence is always monotone in relation to the inclusion operation of sets or subsets of U. Hence $2^U \xrightarrow{w} \mathbb{R}_{\geq 0}$ may, under certain conditions on preferences, have very strong properties making it additive or monotone. These types of utilities are described next.

DEFINITION 3. For any $A, B \subseteq U$, strong utility is defined by the property $w(A \cup B) = w(A) + w(B)$ when $A \cap B = \varnothing$.

THEOREM 1. Let U be the set of all alternatives in a game, $w : 2^U \to \mathbb{R}_{\geq 0}$ be a utility map from the power set of U to the set of all non-negative real numbers $\mathbb{R}_{\geq 0}$, and $A, B \subseteq U$. If utility w is strong, the utility $w(A)$, $A \in 2^U$, is produced from the utility of the singleton sets, i.e. $w(A) = \sum_{\alpha_i \in A} w(\{\alpha_i\})$, $w(\varnothing) = 0$.

Theorem 1 allows us to order totally, just like real numbers, the various gains that result from the choice of players' strategies when these gains are completely arbitrary and thus non-comparable between them. In particular, Theorem 1 states that the utility obtained by each player from each alternative gain is the sum of utilities of the singleton sets that make up each gain tuple. Hence, once we know the utility a player derives from each singleton set, we can obtain the utility of every gain tuple. A detailed numerical example is provided after Theorem 2 and an additional short numerical example is presented in Remark 1 of the Appendix.

DEFINITION 4. (Semi-strong utility) For any $A, B \subseteq U$, semi-strong utility is defined by the property $A \subseteq B$ which implies $w(B) \geq w(A)$. Hence semi-strong utility is monotone.

THEOREM 2. Strong utility, w, is semi-strong, i.e. monotone if $w\left(2^U\right) \subset \mathbb{R}_{\geq 0}$. The condition $w\left(2^U\right) \subset \mathbb{R}_{\geq 0}$ essentially states that in order for the monotonicity property which characterizes semi-strong utility to hold, the utility of each gain tuple, i.e., the utility of each element that belongs to the power set 2^U must be non-negative.

Consider a game with two players,1 and 2. The strategy space of both player 1 and player 2 is the set $S_1 = S_2 = \{s_1, s_2\}$. The universe set of gain alternatives is: $U = \{u_1, u_2, u_3, u_4\}$. Player's 1 soft set is the pair:

$$\left(F_1 :\ \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \rightarrow 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{9}$$

We assume that the map $F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \rightarrow 2^{\{u_1,u_2,u_3,u_4\}}$ is the following:

$$\left\{F_1(s_1, s_1) = \{u_1\},\ F_1(s_1, s_2) = \{u_2\},\ F_1(s_2, s_1) = \{u_2, u_3\},\ F_1(s_2, s_2) = \{u_4\}\right\} \tag{10}$$

Player's 2 soft set is the pair:

$$\left(F_2 :\ \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \rightarrow 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{11}$$

We assume that the map $F_2 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \rightarrow 2^{\{u_1,u_2,u_3,u_4\}}$ for player 2 is the following:

$$\left\{F_2(s_1, s_1) = \{u_1, u_2\},\ F_2(s_1, s_2) = \{u_1, u_3\},\ F_2(s_2, s_1) = \{u_3\},\ F_2(s_2, s_2) = \{u_4\}\right\} \tag{12}$$

The alternative gains in this game are obviously non-totally ordered. We will use the concept of strong utility in order to compute the NE of the game. Let $w : 2^U \rightarrow \mathbb{R}_{\geq 0}$ be the utility map from the power set $2^{\{u_1,u_2,u_3,u_4\}}$ to the closed interval $[0, +\infty) := \mathbb{R}_{\geq 0}$, such that $w(\{u_1\}) = 2,\ w(\{u_2\}) = 7,\ w(\{u_3\}) = 3,$ and $w(\{u_4\}) = 1$. It follows from Theorem 1 that: $w(\{u_2, u_3\}) = 10,\ w(\{u_1, u_3\}) = 5,$ and $w(\{u_1, u_2\}) = 9$. The NE of the game is the combination of strategies (s_2, s_1). Gains are now totally ordered in the length-5 chains:

$$\varnothing < \{u_4\} < \{u_1\} < \{u_2\} < \{u_2, u_3\} \tag{13}$$

and

$$\varnothing < \{u_4\} < \{u_3\} < \{u_1, u_3\} < \{u_1, u_2\} \tag{14}$$

for players 1 and 2, respectively. Hence a game with non-totally ordered gains has now been transformed into a game with totally ordered gains, for which the NE was found to be (s_2, s_1).

One might wonder why the difference between strong and semi-strong utility matters in real-life cases. Strong utility allows players, such as market participants, to rank all outcomes that result from the strategy they choose and pick the one for which their gain is maximized. The result of this process, carried out by each player, is the game's Nash equilibrium. However, the gain obtained from choosing a strategy can be either greater or smaller than the sum of gains that correspond to each element in the relevant gain set. For example, the utility enjoyed by a consumer who consumes products X and Y simultaneously, may be smaller than the sum utility she would have enjoyed, had she consumed these two products separately. To draw an analogy with finance, the gain of an investor who chooses to invest in a portfolio of stocks may be greater or smaller than the sum of gains the investor would have received, had she invested in each stock separately. That's the idea captured by semi-strong utility.

5. Nash equilibrium in mixed strategies (NEMS)

The approach of classical game theory to prove the existence of a Nash equilibrium rests on a set of ideal assumptions. In particular, to prove existence, one must show that the best response correspondence $BR : S \rightrightarrows S$ is closed and convex-valued, where the strategy space S is a nonempty, compact, and convex subset of a finite dimensional normed linear space. Then, by Kakutani's (1941) fixed point theorem, the best response correspondence $BR : S \rightrightarrows S$ has a fixed point $s \in S$ such that $s \in BR(s)$.

The approach of classical game theory has two drawbacks. First, it lacks realism in the following sense: Consider a game with two players, 1 and 2. The best response correspondence of player 1 is $BR_1 : [0, 1] \rightrightarrows [0, 1]$ and similarly for player 2, $BR_2 : [0, 1] \rightrightarrows [0, 1]$. This implies that player 1 has the cognitive ability to calculate her best response for all probabilities contained in the uncountably infinite set $[0, 1]$ with which player 2 plays her strategy. The same holds for player 2. This is very unlikely to be the case in practice.

The second drawback of classical game theory is that it has fewer degrees of freedom compared to the approach of soft set theory. For example, it can only deal with continuous maps defined on convex and compact sets. It does not work for set-valued correspondences defined on discrete sets for which the notion of continuity needs redefinition given that it is not meaningful for such sets, and the usual compact and convex sets must be replaced by a discrete set. To define a Nash equilibrium in such a case, one must make use of a discrete fixed point theorem such as the Iimura et al. (2005) theorem, Yang's (2009) theorem, or Tarski's (1955) lattice fixed point theorem.

To describe NEMS in an abstract manner one needs to extend the strategy space S_1 of player 1 to a superset $\overline{S}_1 \supset S_1$ and similarly extend player 2's strategy space S_2 to a superset $\overline{S}_2 \supset S_2$. The sets $\overline{S}_1$ and $\overline{S}_2$ may contain mixed or other strategies in addition to those contained in the sets S_1 and S_2. In the pure strategy game G, in the language of soft set theory, one has a map $S_1 \times S_2 \to 2^U$. Game G can be extended to include mixed or other alternative strategies. This generates the extended game $\overline{G}$ which is a map $\overline{S}_1 \times \overline{S}_2 \to \overline{2^U}$, where the set $\overline{2^U}$ extends the power set 2^U. The set $\overline{2^U}$ contains mixed gains or further gains than those contained in 2^U and it is ordered either as a lattice or a chain. The restriction of the extended game $\overline{G}$ on the Cartesian product $S_1 \times S_2$ is the initial pure strategies game. In the extended game $\overline{G}$, player 1's best response correspondence is $BR_1(s_2) : \overline{S}_2 \to\to \overline{S}_1$ with $BR_1(s_2) = \underset{s_1 \in \overline{S}_1}{arg\ max}\, \overline{G}(s_1, s_2)$. Players 2's best response correspondence is $BR_2(s_1) : \overline{S}_1 \to\to \overline{S}_2$ with $BR_2(s_1) = \underset{s_2 \in \overline{S}_2}{arg\ max}\, \overline{G}(s_1, s_2)$. The solution to $\overline{G}$ is a pair $\left(s_1^*, s_2^*\right) \in \overline{S}_1 \times \overline{S}_2$ with $s_1^* \in BR_1\left(s_2^*\right)$ and $s_2^* \in BR_2\left(s_1^*\right)$. This generalized approach can accommodate various fixed point theorems depending on the structure of the strategy spaces and the extension $\overline{G}$.

We give an example for clarification purposes by extending a game with no NEMS to a game with NEMS. Consider the following game, in soft set theoretic form, in which there are two players, 1 and 2, with strategy spaces $S_1 = S_2 = \{s_1, s_2\}$. The universe set of alternative gains is $U = \{u_1, u_2, u_3, u_4\}$. Player 1's soft set is the pair:

$$\left(F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{15}$$

for which the map $F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}$ is the following:

$$\left\{F_1(s_1, s_1) = \{u_2\},\ F_1(s_1, s_2) = \{u_1, u_2\},\ F_1(s_2, s_1) = \{u_1, u_2, u_3\},\ F_1(s_2, s_2) = \{u_2\}\right\} \tag{16}$$

Player 2's soft set is the pair:

$$\left(F_2 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}, \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\right) \tag{17}$$

for which the map $F_2 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}$ is the following:

$$\left\{F_2(s_1, s_1) = \{u_1, u_2\},\ F_2(s_1, s_2) = \{u_3\},\ F_2(s_2, s_1) = \{u_1\},\ F_2(s_2, s_2) = \{u_2\}\right\} \tag{18}$$

The above game has no NEMS. We extend player 2's strategy space to the extended strategy space $\overline{S}_2 = \{s_1, s_2, s_3\} \supset S_2$, where $s_3 := \frac{1}{2}s_1 + \frac{1}{2}s_2$. Then, the map $F_1 : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\} \to 2^{\{u_1,u_2,u_3,u_4\}}$ changes to the extended map:

$$\overline{F_1} : \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2), (s_1, s_3), (s_2, s_3)\} \to 2^{\{u_1,u_2,u_3,u_4\}} \cup 2^{\{u_1,u_2,u_3,u_4\}\times\{\frac{1}{2}\}} \tag{19}$$

where

$$\begin{aligned}\{&\overline{F_1}(s_1, s_1) = \{u_2\},\ \overline{F_1}(s_1, s_2) = \{u_1, u_2\},\ \overline{F_1}(s_2, s_1) = \{u_1, u_2, u_3\},\ \overline{F_1}(s_2, s_2) = \{u_2\},\\ &\overline{F_1}(s_1, s_3) = \frac{1}{2}\{u_2\} \cup \frac{1}{2}\{u_1, u_2\},\ \overline{F_1}(s_2, s_3) = \frac{1}{2}\{u_1, u_2, u_3\} \cup \frac{1}{2}\{u_2\}\}\end{aligned} \tag{20}$$

Similarly, for player 2 we have precisely the same general form of the extended map as in Equation (19), where

$$\{\overline{F_2}(s_1,s_1)=\{u_1,u_2\},\ \overline{F_2}(s_1,s_2)=\{u_3\},\ \overline{F_2}(s_2,s_1)=\{u_1\},\ \overline{F_2}(s_2,s_2)=\{u_2\},$$
$$\overline{F_2}(s_1,s_3)=\frac{1}{2}\{u_1,u_2\}\cup\frac{1}{2}\{u_3\},\ \overline{F_2}(s_2,s_3)=\frac{1}{2}\{u_1\}\cup\frac{1}{2}\{u_2\}\} \tag{21}$$

By extending the ordering of the extended gain space such that $\{u_1\},\{u_2\}<\frac{1}{2}\{u_1\}\cup\frac{1}{2}\{u_2\}$, the strategy combination (s_2,s_3) is a NEMS of the extended game.

In classical game theory, a Nash equilibrium in mixed strategies (NEMS) is a probability distribution assigned to each player's strategies such that, each player is indifferent between her pure strategies as they yield the same von Neumann-Morgenstern pay-off. We consider the two player game described in Equations (3)–(6). The strategy space for players 1 and 2 is the set $S_1 = S_2 = \{s_1, s_2\}$ and the universe set of gain alternatives is $U = \{u_1,u_2,u_3,u_4\}$. In contrast to Deli and Çağman (2016a), each player's gains in this example are totally ordered. In particular, for player 1 we have the length-4 chain as described in Equation (7), and similarly for player 2 we have the length-4 chain described in Equation (8), similar to real numbers in classical game theory for which the completeness property is satisfied. This makes the comparison between each player's gains clear and unambiguous. For example, it would be impossible to decide which of the two sets $\{u_1,u_3\}$ and $\{u_2,u_4\}$ is better for a player, given the completely arbitrary nature of u_1,u_2,u_3, and u_4, unless these sets are ordered. From the soft set representation of the game in Equations (3)-(6), we note that $F_1(s_1,s_1)=\{u_1\}$, $F_1(s_1,s_2)=\{u_1,u_2\}$, $F_1(s_2,s_1)=\{u_1,u_2,u_3\}$, and $F_1(s_2,s_2)=\{u_1,u_2,u_3,u_4\}$ and $F_2(s_1,s_1)=\{u_1,u_3\}$, $F_2(s_1,s_2)=\{u_1,u_2,u_3\}$, $F_2(s_2,s_1)=\{u_1\}$, $F_2(s_2,s_2)=\{u_1,u_2,u_3,u_4\}$. Given that the two players' pay-offs are totally ordered, the computation of the NEMS was easily found to be the pair of strategies (s_1,s_2). If no ordering relation exists between arbitrary pay-offs, then even for a simple 2×2 game the computation of the NEMS would not be possible unless we associate a number to each gain, in which case we are back to classical game theory (or use the notions of strong and semi-strong utility of Section 4).

We shall now provide a second approach to the computation of the NEMS which is less abstract compared to that presented at the beginning of the current Section. We know that if $\sigma_1(s_1)\geq 0$ and $\sigma_1(s_2)\geq 0$ stand for the probabilities with which player 1 plays strategies s_1 and s_2, respectively, where $\sigma_1(s_1) + \sigma_1(s_2) = 1$, then a mixed strategy for player 1 is given by the probability distribution $\left\{(s_1, \sigma_1(s_1)),\ (s_2, \sigma_1(s_2))\right\}$. Similarly, if $\sigma_2(s_1)$ and $\sigma_2(s_2)$ stand for the probabilities with which player 2 plays strategies s_1 and s_2, respectively, where $\sigma_2(s_1) + \sigma_2(s_2) = 1$, then a mixed strategy for player 2 is given by the probability distribution $\left\{(s_1, \sigma_2(s_1)),\ (s_2, \sigma_2(s_2))\right\}$.

Since the selection of strategies between players 1 and 2 happens independently, it follows from the definition of statistical independence that the probability with which players 1 and 2 will jointly choose strategy s_1 is equal to the product of their respective probabilities, i.e. $\sigma_1(s_1)\sigma_2(s_1)$. Similarly, the probability that player 1 will choose strategy s_1 and player 2 will choose strategy s_2 is equal to the product of their respective probabilities $\sigma_1(s_1)\sigma_2(s_2)$. With this in mind, the expected gain of player 1, denoted by π_1, is equal to:

$$\pi_1 = \sigma_1(\mathrm{s}_1)\sigma_2(\mathrm{s}_1)\{\mathrm{u}_1\} + \sigma_1(\mathrm{s}_1)\sigma_2(\mathrm{s}_2)\{\mathrm{u}_1, \mathrm{u}_2\} + \sigma_1(\mathrm{s}_2)\sigma_2(\mathrm{s}_1)\{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\} + \sigma_1(\mathrm{s}_2)\sigma_2(\mathrm{s}_2)\{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4\} \tag{22}$$

In the same manner, the expected gain of player 2, π_2, is equal to:

$$\pi_2 = \sigma_1(\mathrm{s}_1)\sigma_2(\mathrm{s}_1)\{\mathrm{u}_1, \mathrm{u}_3\} + \sigma_1(\mathrm{s}_1)\sigma_2(\mathrm{s}_2)\{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\} + \sigma_1(\mathrm{s}_2)\sigma_2(\mathrm{s}_1)\{\mathrm{u}_1\} + \sigma_1(\mathrm{s}_2)\sigma_2(\mathrm{s}_2)\{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4\} \tag{23}$$

To draw an analogy with classical game theory and the concept of NEMS, we recognize each prize u_1, u_2, u_3, and u_4 contained in the universe set $U = \{u_1, u_2, u_3, u_4\}$ as a vector in the standard basis for $\mathbb{R}^4$. Hence we recognize u_1 as being the vector $u_1 :=\langle 1, 0, 0, 0\rangle$, u_2 as being the vector $u_2 :=\langle 0, 1, 0, 0\rangle$, and similarly for u_3 and u_4: $u_3 :=\langle 0, 0, 1, 0\rangle$ and $u_4 :=\langle 0, 0, 0, 1\rangle$. Player's 1 gains $\{u_1, u_2\}$, $\{u_1, u_2, u_3\}$, and $\{u_1, u_2, u_3, u_4\}$ are equal to the unions $\bigcup_{i=1}^{2}\{u_i\}$, $\bigcup_{i=1}^{3}\{u_i\}$, and $\bigcup_{i=1}^{4}\{u_i\}$, respectively. Identifying the union with addition to each strategy we can show that $\{u_1, u_2\}$ is equal to $\bigcup_{i=1}^{2}\{u_i\} = \langle 1, 0, 0, 0\rangle + \langle 0, 1, 0, 0\rangle = \langle 1, 1, 0, 0\rangle$, $\{u_1, u_2, u_3\}$ is equal to $\bigcup_{i=1}^{3}\{u_i\} = \langle 1, 0, 0, 0\rangle + \langle 0, 1, 0, 0\rangle + \langle 0, 0, 1, 0\rangle = \langle 1, 1, 1, 0\rangle$, and gain $\{u_1, u_2, u_3, u_4\}$ is the vector $\bigcup_{i=1}^{4}\{u_i\} = \langle 1, 0, 0, 0\rangle + \langle 0, 1, 0, 0\rangle + \langle 0, 0, 1, 0\rangle + \langle 0, 0, 0, 1\rangle = \langle 1, 1, 1, 1\rangle$ with all entries equal to 1.

Similarly, for player 2 the gain $\{u_1, u_3\}$ is the union $\{u_1\} \cup \{u_3\}$ or the vector $\langle 1, 0, 0, 0\rangle + \langle 0, 0, 1, 0\rangle =$

$\langle 1,0,1,0\rangle$. Hence the expected gain of player 1, π_1, becomes:

$$\pi_1 = \sigma_1(\mathrm{s}_1)\,\sigma_2(\mathrm{s}_1)\,\langle 1,0,0,0\rangle + \sigma_1(\mathrm{s}_1)\,\sigma_2(\mathrm{s}_2)\,\langle 1,1,0,0\rangle +$$
$$\sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_1)\,\langle 1,1,1,0\rangle + \sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_2)\,\langle 1,1,1,1\rangle$$

or

$$\pi_1 = \langle 1, \sigma_1(\mathrm{s}_1)\,\sigma_2(\mathrm{s}_2) + \sigma_1(\mathrm{s}_2)\,, \sigma_1(\mathrm{s}_2)\,, \sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_2)\rangle \tag{24}$$

The expected gain for player 2 computed in the same manner is given by the expression:

$$\pi_2 = \sigma_1(\mathrm{s}_1)\,\sigma_2(\mathrm{s}_1)\,\langle 1,0,1,0\rangle + \sigma_1(\mathrm{s}_1)\,\sigma_2(\mathrm{s}_2)\,\langle 1,1,1,0\rangle +$$
$$\sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_1)\,\langle 1,0,0,0\rangle + \sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_2)\,\langle 1,1,1,1\rangle$$

or

$$\pi_2 = \langle 1, \sigma_2(\mathrm{s}_2)\,, \sigma_1(\mathrm{s}_1) + \sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_2)\,, \sigma_1(\mathrm{s}_2)\,\sigma_2(\mathrm{s}_2)\rangle \tag{25}$$

We define the Nash equilibrium in mixed strategies as the probabilistic mix

$$\left(\left(\sigma_1^*(s_1)\,, \sigma_1^*(s_2)\right), \left(\sigma_2^*(s_1)\,, \sigma_2^*(s_2)\right) \right) \tag{26}$$

which maximizes simultaneously the convex combination of the elements in the expected gain vectors π_1 and π_2 of players 1 and 2, respectively. Hence the Nash equilibrium in mixed strategies corresponds to the solution of the following two maximization problems:

$$\underset{\sigma_1(s_1)}{max}\, w_1 + w_2\left(\sigma_1(s_1)\,\sigma_2(s_2) + \sigma_1(s_2)\right) + w_3\sigma_1(s_2) + w_4\sigma_1(s_2)\,\sigma_2(s_2) \tag{27}$$

and

$$\max_{\sigma_2(s_1)} \theta_1 + \theta_2\sigma_2(s_2) + \theta_3(\sigma_1(s_1) + \sigma_1(s_2)\sigma_2(s_2)) + \theta_4\sigma_1(s_2)\sigma_2(s_2) \tag{28}$$

such that

$w_i \geq 0, \theta_i \geq 0, i = 1, 2, 3, 4, w_1 + w_2 + w_3 + w_4 = 1$, and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1$.

6. Cooperative bargaining games

This section discusses cooperative bargaining games using soft set theory. Section 6.1 describes the Nash bargaining solution offering a different perspective based on soft set theory. Section 6.2 presents an application to bargaining in over-the-counter financial markets. Section 6.3 offers a numerical example of a dynamic search-and-bargaining game in over-the-counter financial markets, motivated by the work of Duffie et al. (2005).

6.1. The Nash bargaining solution

In cooperative bargaining games the outcomes are binding by definition. An example of a cooperative bargaining game is the wage setting between a labor union and an employer. The outcome of this game is a legally binding contract between the two players. Von Neumann and Morgenstern (1944) defined the bargaining set as the set of all individually rational and Pareto-efficient pairs [7]. The outcome of the bargaining will lie within this set. By introducing the axioms of symmetry, invariance to equivalent utility

[7] The impact of the works of John von Neumann and Oskar Morgenstern on the scientific community and the further development of social sciences has been tremendous. Their theoretical contributions were too mathematical for economists, thus game theory was developed almost entirely by mathematicians of that period. As Hanappi (2013) explains, since von Neumann and Morgenstern's pioneering contributions, game theory has had a mixed fate, with periods of ignorance and research inactivity changing with periods of redirection towards new fields of interest. Significant subsequent contributions to von Neumann and Morgenstern's cooperative theory include those of Nash (1950,1951,1953) and Shapley (1953) who laid the groundwork for non-cooperative theory, cooperative bargaining theory, and the theory of stochastic games. Kuhn (1950) introduced the concepts of behavior strategies while Albert Tucker set the stage for the further development of the interplay between competition and cooperation. Nevertheless, other scientific fields were also influenced by von Neumann-Morgenstern. Shannon and Weaver's (1949) work on cryptographic military methods was massively influenced by von Neumann and Morgenstern's work, introducing an engineering perspective in game theoretical problems. Wiener (1948, 1954) proposed to look into "black boxes", a term used to refer to processes with a rigid engineering attitude, to turn them into "white boxes", i.e. the explicit statement of a full-fledged equation system or program, as per von Neumann and Morgenstern. A number of influential ideas on how order can emerge out of randomness, such as those of Prigogine (1984) and Kaufman (1993) were also influenced by the early contributions of von Neumann and Morgenstern. Prigogine's (1984) contribution was in chemistry and showed that living systems are characterized by processes that deviate from thermodynamic equilibrium. Another contribution came from biology (Maynard-Smith, 1982, 1988) with researchers inspired by game-theoretic modelling techniques that enabled them to depart from standard mathematical solutions. Computer science was also influenced by von Neumann and Morgenstern which led to

representations, and independence of irrelevant alternatives (IIA), Nash (1950) derived the Nash bargaining solution as the point (x, y) in the bargaining set, which yields the maximum value of the Nash product $[u(x) - u(d)][v(y) - v(d)]$ subject to the constraint $x + y = 1$, where u and v are the two players' utility functions, and the real numbers $u(d)$ and $v(d)$ are their respective utilities in the event that no agreement is reached.[8] The constraint $x + y = 1$ indicates that the sum of the shares received by the two players must sum up to 1, i.e. the entire prize of the bargaining process.

Let U denote the universe set containing all possible alternatives for each player in a cooperative bargaining game. In a soft set theory setting, the players' gains in the event of no agreement correspond to one or more elements in the power set 2^U. As an example, let $U = \{u_1, u_2, u_3, u_4\}$ and let the sets $P_1 = \{u_2\}$ and $P_2 = \{u_1, u_3\}$ represent player's 1 and 2 minimally required gains. It follows that player 1 and player 2 will walk off from the bargaining process for anything less. The upper set of P_1 is defined as the set whose elements are the supersets of P_1. Similarly, for the upper set of P_2. Hence in this example, the upper set of P_1 is defined as the set:

$$C_1 := \Big\{\{u_2\}, \{u_1, u_2\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_1, u_2, u_3\}, \{u_1, u_2, u_4\}, \{u_2, u_3, u_4\}, \{u_1, u_2, u_3, u_4\}\Big\} \tag{29}$$

while the upper set of P_2 is defined as the set

$$C_2 := \Big\{\{u_1, u_3\}, \{u_1, u_2, u_3\}, \{u_1, u_2, u_3, u_4\}\Big\} \tag{30}$$

If A_1 and A_2 are elements of C_1 and C_2, respectively, the Nash bargaining solution is defined as the union $A_1 \cup A_2$, such that $A_1 \cup A_2 = U$. Hence the Nash bargaining solution within the framework of soft set theory consists of the elements in the players' upper sets, such that each player receives her optimal gain and $A_1 \cup A_2 = U$.

This problem may have more than one solution or a unique solution. In any case, the players' sets which correspond to their minimally required gains in the event of no agreement, must be disjoint as no

the advancement of evolutionary economics (Nelson and Winter, 1982). For a historical account of the developments in game theory see Shubik (2011) and Weintraub (1992).

[8] Nash (1953) extended his previous treatment of the bargaining problem to a wider class of situations in which threats can play a role in negotiations.

two players can gain the same prize simultaneously, unless this alternative has multiplicity greater than one. Considering $U = \{u_1, u_2, u_3, u_4\}$, $P_1 = \{u_2\}$, and $P_2 = \{u_1, u_3\}$, one solution to the Nash bargaining problem is $A_1 = \{u_2, u_4\} \in C_1$ and $A_2 = \{u_1, u_3\} \in C_2$, with $A_1 \cup A_2 = U$. The only case the solution is unique is when A_1 and A_2 are the minimally required sets by players 1 and 2, respectively, such that their union is the set U. For example, if P_1 and P_2 denote the sets $\{u_1, u_2\}$ and $\{u_3, u_4\}$ satisfying $\{u_1, u_2\} \cup \{u_3, u_4\} = U$, there exists a unique solution to the bargaining problem.

When more than one solution exist, choosing the optimal solution requires the existence of an ordering rule. If the axiom of symmetry or anonymity is dropped out, so that the labeling of players matters, we obtain the generalized Nash bargaining solution which is the point (x, y) in the bargaining set, in which the weighted utility $[u(x) - u(d)]^a [v(y) - v(d)]^b$ is maximum. This is subject to the constraint condition $x + y = 1$, where a and b are measures of the players' bargaining power usually summing up to 1, reflecting the relative impatience to conclude the bargaining. In a soft set setting, this is translated as attaching differing weights on the elements of the universe set U, where each element has different significance level for each player.

6.2. An application to dynamic search and bargaining in over-the-counter (OTC) markets

Motivated by the work of Duffie et al. (2005) we translate a problem of dynamic search and bargaining equilibrium in over-the-counter (OTC) markets with competing marketmakers into a soft set theory problem. In OTC financial markets an investor who wishes to sell a particular asset must search for a buyer (Vayanos and Wang, 2007; Lagos et al., 2011). This implies opportunity or other costs which, in turn, open the way for market intermediaries. In such markets, counterparties must act strategically as prices are set through a bargaining process that reflects each investor's ability to trade, taking into account inventory and transaction costs, as well as various trading frictions that arise through bargaining.

Similar to Duffie et al. (2005) we assume that there is no inventory risk among marketmakers because of the existence of interdealer markets. Interdealer markets allow marketmakers to liquidate their positions instantly, thus marketmakers have no inventory risk at any time. We also make the assumption that agents in our model are symmetrically informed. That is, our analysis assumes no market frictions exist in the market-place, and marketmakers' bid and ask prices are not explained by microstructure inventory considerations

(Garman, 1976; Stoll, 1978; Amihud and Mendelson, 1980; Ho and Stoll, 1980, 1981, 1983; Kyle, 1989; Jankowitsch et al., 2011; Friewald and Nagler, 2019; Colliard et al., 2021) or by adverse selection considerations arising from asymmetric information (Glosten and Milgrom, 1985; Kyle, 1985; Easley and O'Hara, 1987; O'Hara, 1995; Calcagno and Lovo, 2006; Glode and Opp, 2016; Ranaldo and Somogyi, 2021). It follows that the bid and ask prices are set based on the market participants' ability to find counterparties to transact with. It is further assumed that investors are homogeneous with respect to the speed with which they find counterparties.

Typical examples of OTC markets in which asymmetric information is absent (or limited), include the foreign exchange market and the interest-rate swaps market. In Duffie et al. (2005) marketmakers compete with each other for order flow and liquidity and as a result they quote better prices. In fact, they are forced to do so as investors have the option to shop around in search of better deals. When investors' search alternatives for suitable counterparties improve, and when marketmakers' contact intensities become larger, bid-ask spreads tend to vanish, provided that marketmakers do not possess full bargaining power (a monopolistic marketmaker case).

We let time t be discrete, taking values in the countably infinite time set $T{:=}\{0, 1, 2, ...\}$. Investors are divided into four categories according to an intrinsic type labeled *'high' (h)* or *'low' (l)*, and their status as owners *(o)* or not *(n)* of an asset. Thus the full set of investor types denoted by τ, is defined as $\tau{:=}\{ho,\ hn,\ lo, ln\}$. Low-type investors who own an asset bear holding costs, whilst high-type investors do not bear such costs. We define low-type investors as those with low liquidity, high financing costs, or a tax disadvantage over high-type investors. In a dynamic search-and-bargaining equilibrium, low-type investors who own an asset are those who take the sell-side of a transaction, whereas high-type investors who do not own an asset are those who take the buy-side of the transaction. This is actually the only case a gain from a transaction may arise, regardless of whether investors have private information about their own type.

Let N be a positive integer. The universe set $U(t) = \{c_1(t), ..., c_N(t)\}$ consists of N consumption alternatives expressed in monetary terms. As described in the previous section, at each time t, each investor belonging to each of the four categories chooses a consumption plan from her upper set, such that the union of all investors' consumption plans is equal to the universe set $U(t)$. Each investor is assumed to have a minimally required set of consumption gains. The Nash bargaining equilibrium at time t is a quadruple

$\{P(t), B(t), A(t), M(t)\}$ which consists of the price $P(t)$ negotiated directly between investor groups *lo* and *hn*, the bid price $B(t)$ at which investors sell to marketmakers, the ask price $A(t)$ at which investors buy from marketmakers, and the interdealer price $M(t)$.

Accordingly, let $P_{ho}(t)$ be the minimally required set of consumption gains for a type *ho* investor, and similarly let $P_{hn}(t)$, $P_{lo}(t)$, and $P_{ln}(t)$ correspond to the minimally required set of consumption gains for investor types *hn*, *lo*, and *ln*, respectively. Furthermore, let $C_{ho}(t)$, $C_{hn}(t)$, $C_{lo}(t)$, and $C_{ln}(t)$ denote the respective upper sets for each investor. At time t, each investor belonging to each of the four categories selects an element from her upper set. The union of all these elements must equal the universe set $U(t)$.

If we denote by $S_{ho}(t)$, $S_{hn}(t)$, $S_{lo}(t)$ and $S_{ln}(t)$ the elements in the respective upper sets selected by each of the four investor types and let numbers A_1, A_2, A_3, and A_4 be the sums of the elements contained in $S_{ho}(t)$, $S_{hn}(t)$, $S_{lo}(t)$ and $S_{ln}(t)$, respectively, a Nash (1950) bargaining with a seller bargaining power of $\theta \in [0, 1]$ yields:

$$P(t) = (A_3 - A_4)(1 - \theta) + (A_1 - A_2)\theta \tag{31}$$

This equation is the analogue of Equation (11) in Duffie et al. (2005) expressed in a language of soft set theory. Bid and ask prices are determined in a similar way through bargaining between investors and marketmakers. Marketmakers have a fraction $z \in [0, 1]$ of bargaining power when they negotiate with investors. Hence a marketmaker buys from an investor at the bid price $B(t)$ and sells at the ask price $A(t)$ determined as:

$$A(t) = (A_1 - A_2)z + M(t)(1 - z) \tag{32}$$

$$B(t) = (A_3 - A_4)z + M(t)(1 - z) \tag{33}$$

Assuming that marketmakers meet more potential buyers than sellers, the interdealer price $M(t)$ is equal to the ask price $A(t)$, whereas if marketmakers meet more potential sellers than buyers the interdealer price is equal to the bid price $B(t)$. If the number of potential buyers is equal to that of potential sellers, the interdealer price is equal to:

$$M(t) = \overline{q}(A_1 - A_2) + (1 - \overline{q})(A_3 - A_4) \quad (34)$$

for $\overline{q} \in [0, 1]$ arbitrary. This is the equivalent to the knife-edge case discussed in Duffie et al. (2005).

6.3. Numerical example

Further to the problem of dynamic search and bargaining equilibrium discussed in the previous section, we provide a numerical example which assumes a competitive Walrasian equilibrium framework, according to which, investors transact instantly with one another and supply equals demand at every point in time. In such framework, prices approach the competitive Walrasian prices and bid-ask spreads approach zero as investors find each other in the marketplace more quickly. Fast intermediation by competing marketmakers also results in competitive Walrasian prices and zero bid-ask spreads. The following numerical example is not applicable to the case of a fast monopolistic marketmaker whose intermediation does not lead to competitive Walrasian prices and narrower bid-ask spreads, but leads to wider bid-ask spreads as intermediation increases.

Let the universe set be $U(t) = \{100, 150, 180, 50, 80, 280, 130, 120, 70\}$ containing N=9 alternative consumption plans expressed in monetary terms. Investors who are sellers are assumed to have bargaining power $\theta = 0.5$ and competing marketmakers are assumed to have bargaining power $z = 0.2$. Let the minimally required consumption sets contained in the set of investor types $\tau := \{ho, hn, lo, ln\}$, be $P_{ho}(t) = \{100\}$, $P_{hn}(t) = \{50\}$, $P_{lo}(t) = \{120\}$, and $P_{ln}(t) = \{80\}$, respectively. There still remain five elements from the universe set $U(t)$ to be distributed to the four investors, i.e. $\{150, 180, 280, 130, 70\}$.

In number theory and combinatorics, an integer partition provides a way of writing a positive integer as a sum of positive integers. The positive integer 5, which corresponds to the number of elements in the set $\{150, 180, 280, 130, 70\}$ can be expressed as a sum of positive integers. If the ordering of the summands in that sum of positive integers is of no importance, the positive integer 5 can be partitioned in seven different ways, i.e. 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1. As we are interested in partitions that contain four summands corresponding to the number of investor categories, we exclude the last partition. Using discrete mathematics and combinatorial analysis, the number of ways through which we can partition a set of n objects into r cells, with $n_1 \geq 0$ elements of a first kind in the first cell, $n_2 \geq 0$ elements of a second kind in the second cell, and so forth, is given by the multinomial coefficient

$$\binom{n}{n_1,\ n_2,\ ...,n_r} = \frac{n!}{n_1!\ n_2!...n_r!}$$

where $n_1 + n_2 + ... + n_r = n$.

Trivially, there is only one way to partition five elements into four cells where three contain no elements and the fourth contains all five elements. Given that there are four investors, each investor may get all five elements whilst the rest of investors may get zero. Hence there are four different ways through which the five consumption gains expressed in monetary terms in the set $\{150, 180, 280, 130, 70\}$ can be distributed to the four investors. There are five partitions of $\{150, 180, 280, 130, 70\}$ into four cells where the first cell contains four distinct elements, the second cell contains one element, and the remaining two cells are equal, both containing zero elements. This result is obtained using the multinomial coefficient with $n = 5,\ n_1 = 4,\ n_2 = 1$, and $n_3 = n_4 = 0$: $\binom{5}{4,1,0,0} = \frac{5!}{4!1!0!0!} = 5$, with $4 + 1 + 0 + 0 = 5 = n$. Using combinatorial analysis, the number of permutations with n objects of which a are alike, b are alike, and so forth, is $\frac{n!}{a!b!...}$. Hence the number of permutations of the four cells in this case is $\frac{4!}{2!} = 12$, given that there are $2! = 2$ cells which are exactly the same and both contain zero elements. It follows that there are twelve different ways to distribute this particular partition among the four investors. Given the number of distinct partitions is five, there are sixty ways to distribute them to the four investors.

Similarly, there exist ten distinct partitions of the set $\{150, 180, 280, 130, 70\}$ into four cells where the first cell contains three elements, the second cell contains two elements, and the remaining two cells contain zero elements each. One can calculate the remaining partitions of this set and the number of ways to distribute them to the four investors in the same manner. The numbers $A_1,\ A_2,\ A_3$, and A_4 in Equations (32)-(34) are functions of a particular partition each time. This implies that the ask and bid prices are also functions of that partition. If we denote the partitions by δ, the ask and bid prices are functions, f_1 and f_2, and take the general form $A = f_1(\delta)$ and $B = f_2(\delta)$.

Our objective is to locate a partition which minimizes the bid-ask spread, i.e. it solves the following mathematical optimization problem:

$$min f_1(\delta) - f_2(\delta)$$

$$s.t.\ \delta \in \Delta \tag{35}$$

where Δ is the set of all partitions. Inserting Equation (34) into Equations (32) and (33), yields Equations (36) and (37) shown below:

$$A(t) = \{z + \overline{q}(1-z)\}(A_1 - A_2) + (1-\overline{q})(1-z)(A_3 - A_4) \tag{36}$$

$$B(t) = \overline{q}(1-z)(A_1 - A_2) + \{z + (1-\overline{q})(1-z)\}(A_3 - A_4) \tag{37}$$

Therefore, the spread is given by the following equation:

$$A(t) - B(t) = z(A_1 + A_4 - A_2 - A_3) \tag{38}$$

For $z = 0.2$ and $\overline{q} = 0.8$, the bid-ask spread is equal to:

$$A(t) - B(t) = 0.2(A_1 + A_4 - A_2 - A_3) \tag{39}$$

We solve the following integer linear programming (ILP) problem:

$$\min_{A_1,A_2,A_3,A_4} (A_1 + A_4 - A_2 - A_3)$$

$$s.t.\ A_1 + A_4 - A_2 - A_3 \geq 0 \tag{40}$$

$$\phi \in \Phi$$

where the factor 0.2 can be omitted. The constraint $A_1 + A_4 - A_2 - A_3 \geq 0$ expresses the condition that the bid-ask spread is non-negative. The second constraint indicates that the distribution we are searching for lies within the set Φ of all distributions between the two investor groups.

The integer linear programming model in Equation (40) is a simplification of the computationally intensive problem in Equation (35). We have split investors into two distinct groups rather than considering each investor separately. The ILP problem in Equation (40) is equivalent to minimizing the sum $A_1 + A_4$ and maximizing the sum $A_2 + A_3$. The two investor types *ho* and *ln* start with an initial combined consumption gain of 100+80=180 which is the sum of their minimally required consumption gains $P_{ho}(t) = \{100\}$ and $P_{ln}(t) = \{80\}$, respectively. Similarly, the two investor types *hn* and *lo* start with an initial combined consumption gain of 50+120=170 which is the sum of their minimally required consumption gains $P_{hn}(t) = \{50\}$ and $P_{lo}(t) = \{120\}$, respectively. It follows that the number of ways of selecting two consumption gains out of five is equal to the number of combinations of five distinct objects taken two at a time, that is, ${}_5C_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$. One way to achieve this is to assign to the first group, which is comprised of the two investor types *ho* and *ln*, the consumption gains 150, 180, and 70. Correspondingly, we can assign to the second group, which is comprised of the two investor types *hn* and *lo*, the consumption gains 280 and 130. In doing so, we get $(A_1 + A_4) = (A_2 + A_3) = 580$, which implies that the bid-ask spread achieves its absolute minimum, i.e. it equals zero.

Tables 1 and 2 present various scenarios and the corresponding outcomes. In Table 1, the first scenario shown in the third row and first column, illustrates the case of an investor who gets only her minimally required consumption gain and nothing else. As explained previously, the numbers A_1, A_2, A_3 and A_4 are the sums of the elements which are minimally required by each investor of type *ho*, *hn*, *lo*, *ln*, respectively, plus one or more gains from the set of remaining consumption gains $\{150, 180, 280, 130, 70\}$. In the fourth row and first column of Table 1 the investor type *ho* gets her minimal requirement, i.e. 100, plus two additional gains from the set of consumption gains $\{150, 180, 280, 130, 70\}$, and in particular, consumption gains 180 and 70. The investor type *hn* shown in the second column and fourth row of Table 1, gets her minimally required consumption gain, i.e. 50 plus one additional gain from the set $\{150, 180, 280, 130, 70\}$, and in particular, the consumption gain 280. The same process is repeated for investor types *lo* and *ln*. The remaining calculations are performed in a similar manner.

Panel A of Table 2 displays the scenarios discussed in Table 1, whilst Panel B shows the solutions for each scenario. To compute the interdealer price we apply directly Equation (34) with $\overline{q} \in [0, 1]$. As an example, the interdealer price 48 in the third row is computed as follows: $M(t) = 0.8(100-50)+0.2(120-80) = 48$, where we have chosen $\overline{q} = 0.8$. The ask price, the bid price, the bid-ask spread, and the price of the asset are computed using Equations (36), (37), (38), and (31), respectively. For example, the ask price 48.4 is computed as follows:

$$A(t) = \{z + \overline{q}(1-z)\}(A_1 - A_2) + (1-\overline{q})(1-z)(A_3 - A_4) =$$
$$\{0.2 + 0.8(1-0.2)\}(100-50) + (1-0.8)(1-0.2)(120-80) = 48.4$$

Correspondingly, the bid price, the bid-ask spread, and the price of the asset are computed as follows:

$$B(t) = \overline{q}(1-z)(A_1 - A_2) + \{z + (1-\overline{q})(1-z)\}(A_3 - A_4) =$$
$$0.8(1-0.2)(100-50) + \{0.2 + (1-0.8)(1-0.2)\}(120-80) = 46.4$$

$$A(t) - B(t) = z(A_1 + A_4 - A_2 - A_3) = 0.2(100+80-50-120) = 2$$

$$P(t) = (A_3 - A_4)(1-\theta) + (A_1 - A_2)\theta = (120-80)(1-0.5) + (100-50)0.5 = 45$$

In the rest of scenarios displayed in Table 2, the bid-ask spread decreases to zero. This result is in line with the findings in Duffie et al. (2005) and corresponds to the cases of a fast investor and a competing marketmaker in which, prices become competitive and the bid-ask spread approaches zero as investors find each other more quickly and marketmakers compete with each other for faster intermediation. It must be noted that in the case of a fast monopolistic marketmaker, fast intermediation will not lead to competitive prices and subsequently to vanishing bid-ask spreads, but will result in wider bid-ask spreads as intermediation by marketmakers increases. The latter result would hold for a monopolistic marketmaker with $z = 1$ where the bid-ask spread increases in the intensity of meeting dealers and thus does not approach zero. Clearly, covering all possible scenarios discussed in Duffie et al. (2005) lies outside the scope of this study.

[Table 1 about here.]

[Table 2 about here.]

7. Conclusion

In this paper we discuss some solution concepts of game theory using soft set theory. In order to consistently align classical game theory with soft set theory in relation to a game's NE, the notion of homogenizing players' gains is introduced. To this end two assumptions are made. Based on the first assumption, players' gain tuples must be totally ordered in chains of various lengths according to the exact format of the game. If this assumption is relaxed, it would be hard to understand the mechanics of the game and identify how strategies are chosen given the completely arbitrary nature of alternatives available to each player. The second assumption regards rejecting bads, that is, points in a player's gain tuple that decrease the player's pay-off. If this assumption is also relaxed, the total ordering of gains would still not work with respect to finding a game's solution, i.e. its NE.

In a second step, we introduce the concepts of strong and semi-strong utility. These concepts build upon the framework of a utility correspondence whose image is the set of all non-negative real numbers and whose objective is to assign such numbers to all alternatives, making them comparable to one another similar to classical game theory. The strong and semi-strong utility has allowed us to convert non-totally ordered gains into totally ordered ones and thus compute the game's NE unambiguously.

We then examine the concept of NEMS. We start with a general framework that gives rise to an extended game which involves the players' strategy spaces and the game's pure strategies power set, ordered either as a lattice or a chain. We then define the best response correspondences in this setup. Finally, we present an application of soft set theory to cooperative bargaining games in over-the-counter (OTC) financial markets. The solution of such a game is defined as a pair of upper set elements, one for each player, such that their union is equal to the universe set. We point out that the solution need not be unique. If this is indeed the case, then choosing the optimal solution would require the introduction of an ordering relation for the various solutions. A solution is guaranteed to be unique if the upper set elements, whose union is the universe set, coincide with those of the minimally required sets by each player, provided that negotiations will continue. Along these lines, a numerical example of a dynamic search-and-bargaining game in an OTC financial market is presented motivated by the work of Duffie et al. (2005). In this numerical example, we compute the price of an asset, the bid and ask price, as well as the bid-ask spread for different types of scenarios by solving an integer linear programming problem.

Game theory is a fascinating research area which has been developed as a stand-alone field of economics. Since the seminal works of Von Neumann (1928) and Von Neumann and Morgenstern (1944) much progress has been made in applying game theoretic models to a wide range of economic problems. On the other hand, soft set theory is a new research area proposed in the late nineties and its study as a useful tool for explaining solution concepts in game theory is still in the early stages. We hope that our findings will provide avenues for future research in this area.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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Appendix - Proofs of Theorems

Proof of Theorem 1. If we define $w(A) = \sum_{\alpha_i \in A} w(\{\alpha_i\})$ and $w(\emptyset) = 0$ this in turn defines a map $w : 2^U \to \mathbb{R}_{\geq 0}$ which satisfies the properties of strong utility. Conversely, if we have a utility map $w : 2^U \to \mathbb{R}_{\geq 0}$ with $w(A \cup B) = w(A) + w(B)$ when $A \cap B = \emptyset$, then $w(\{\alpha_i\})$ exists in $\mathbb{R}_{\geq 0}$ and $w\left(\left\{\alpha_i, \alpha_j\right\}\right) = w\left(\{\alpha_i\} \cup \left\{\alpha_j\right\}\right) = w(\{\alpha_i\}) + w\left(\left\{\alpha_j\right\}\right) - w\left(\{\alpha_i\} \cap \left\{\alpha_j\right\}\right) = w(\{\alpha_i\}) + w\left(\left\{\alpha_j\right\}\right)$. Inductively, this is true for sets with more than two elements.

Remark 1. Theorem 1 implies that strong utility is additive. For example, if the utility obtained from some alternative $\{u_1\}$ is equal to 2, and the utility obtained from some other alternative $\{u_2\}$ is 3, then the utility of the alternative $\{u_1, u_2\}$ is equal to 5, where $\{u_1\}, \{u_2\} \subseteq \{u_1, u_2\}, \{u_1\} \cap \{u_2\} = \emptyset$ and $w(\emptyset) = 0$.

Proof of Theorem 2. Let w be strong and $w\left(2^U\right) \subset \mathbb{R}_{\geq 0}$. If $A \subseteq B$, then $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint. By the strong utility property, $w(B) = w(A) + w(B \backslash A)$, where $w(B \backslash A) \geq 0$ implies $w(B) \geq w(A)$.

Remark 2. Theorem 2 implies that two alternatives may fit very well together, indicating that the utility earned by a player is greater than the utility of each alternative taken separately. Alternatively, it may indicate that two alternatives may not fit well together, i.e. the sum utility gained is less than the sum of utilities taken separately. If for example $w(\{u_1\}) = 2$ and $w(\{u_2\}) = 3$, then $w(\{u_1, u_2\}) \geq 3$. If the two alternatives don't fit well together, then $w(\{u_1, u_2\}) = 3.5$ which is less than the sum utility $w(\{u_1\}) + w(\{u_2\}) = 5$, whereas if one alternative complements another it may be that $w(\{u_1, u_2\}) = 6$.

Table 1: Numerical bargaining scenarios. The table presents bargaining scenarios for four investor types: *ho*, *hn*, *lo*, and *ln*. The investors are categorised according to an intrinsic type labeled 'high' (*h*) or 'low' (*l*), and their status as owners (*o*) or not (*n*) of an asset. Low-type investors who own an asset bear holding costs, whilst high-type investors do not bear such costs. The numbers A_1, A_2, A_3, and A_4 denote the sums of the elements which are minimally required by each investor of type *ho*, *hn*, *lo*, *ln*, respectively, plus one or more gains from a set of remaining consumption gains.

Bargaining scenarios			
Investor type: *ho*	Investor type: *hn*	Investor type: *lo*	Investor type: *ln*
A_1 = 100 + 0	A_2 = 50 + 0	A_3 = 120 + 0	A_4 = 80 + 0
A_1 = 100 + 180 + 70	A_2 = 50 + 280	A_3 = 120 + 130	A_4 = 80 + 150
A_1 = 100 + 150 + 70	A_2 = 50 + 130	A_3 = 120 + 280	A_4 = 80 + 180
A_1 = 100 + 180	A_2 = 50 + 130	A_3 = 120 + 280	A_4 = 80 + 150 + 70
A_1 = 100 + 150	A_2 = 50 + 131	A_3 = 120 + 280	A_4 = 80 + 180 + 70

Table 2: Numerical bargaining scenarios and solutions. Panel A of the table presents bargaining scenarios for four investor types: *ho*, *hn*, *lo*, and *ln* while Panel B shows the solutions for each scenario. The investors are categorised according to an intrinsic type labeled 'high' (*h*) or 'low' (*l*), and their status as owners (*o*) or not (*n*) of an asset. Low-type investors who own an asset bear holding costs, whilst high-type investors do not bear such costs. The numbers A_1, A_2, A_3, and A_4 denote the sums of the elements which are minimally required by each investor of type *ho*, *hn*, *lo*, *ln*, respectively, plus one or more gains from a set of remaining consumption gains. $M(t)$, $A(t)$, $B(t)$ denote the interdealer price, ask price, and the bid price, respectively. $A(t) - B(t)$ denotes the bid-ask spread, while $P(t)$ is the asset price.

Panel A: Bargaining scenarios				
Investor type: *ho*	Investor type: *hn*	Investor type: *lo*	Investor type: *ln*	
A_1	A_2	A_3	A_4	
100	50	120	80	
350	330	250	230	
320	180	400	260	
280	180	400	300	
250	180	400	330	
Panel B: Solutions				
Interdealer price $M(t)$	Ask price $A(t)$	Bid price $B(t)$	Bid-Ask spread $A(t) - B(t)$	Asset price $P(t)$
48	48.4	46.4	2	45
20	20	20	0	20
140	140	140	0	140
100	100	100	0	100
70	70	70	0	70