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Single-name concentration risk in credit portfolios: a comparison of concentration indices

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Single-name concentration risk in credit portfolios: a comparison of concentration indices

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Abstract For assessing the effect of undiversified idiosyncratic risk, Basel II has established that banks should measure and control their credit concentration risk. Concentration risk in credit portfolios comes into being through an uneven distribution of bank loans to individual borrowers (single-name concentration) or through an unbalanced allocation of loans in productive sectors and geographical regions (sectoral concentration). To evaluate single-name concentration risk in the literature concentration indices proposed in welfare (Gini Index) and monopoly theory (Herfindahl-Hirschman index, Theil entropy index, Hannah-Kay index, Hall-Tidemann index) have been used. In this paper such concentration indices are compared by using as benchmark six properties that ensure a consistent measurement of single-name concentration. Finally, the indices are compared on some portfolios of loans.

1 Introduction

The Asymptotic Single-Risk Factor (ASRF) model (Gordy, 2003) that underpins the Internal Rating Based (IRB) approach in the Basel II Accord (Basel Committee on Banking Supervision, BCBS 2004) assumes that idiosyncratic risk has been diversified away fully in the portfolio, so that economic capital depends only on systematic risk contributions. Systematic risk represents the effect of unexpected changes in macroeconomic and financial market conditions on the performance of borrowers. On the other hand, idiosyncratic risk represents the effects of risks that are particular to individual borrowers. In order to include idiosyncratic risk in economic capital, Basel II (BCBS, 2004) requires that banks estimate concentration risk. Concentration risks in credit portfolios arise from an unequal distribution of loans to

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single borrowers (*single-name concentration*) or industrial or regional sectors (*sector concentration*). This paper is focused only on the single-name concentration, in particular in the context of loan portfolios.

Five concentration measures, which have been proposed in welfare and monopoly theory, are compared as regards six desirable properties for measurements of single-name concentration risk. The first index is the Gini coefficient, a widely applied inequality measure. To understand the difference between inequality and concentration, a portfolio that contains few large exposures is considered. By including in such portfolio many small exposures so that even in aggregate their share of the portfolio exposure is very low. Therefore, concentration has not been significantly affected, but the degree of inequality in borrowers' exposures has greatly increased. It follows that inequality measures are very sensitive to the number of small exposures.

Some other indices (the Hannah-Kay index, the Herfindahl-Hirschman index, the Hall-Tidemann index) have been proposed in monopoly theory, and one (the Theil index) arises from the information theory. Single-name concentration risk, analogously to industrial concentration, is due to both a small number of loans and high credit exposures in a portfolio. It follows that these indices, unlike the Gini index, depend on the number of loans in portfolio. By normalizing these indices, the important information of the number of loans in the portfolio would be lost, so the normalization is not applied to them in this paper.

An interesting theoretical result of this work is that the Hannah-Kay index, the Herfindahl-Hirschman index and the Hall-Tidemann index satisfy all the six desirable properties for measurements of single-name concentration risk. In order to compare the features of the concentration indices set up by the six properties, six portfolios with different levels of single-name concentration risk are analyzed. The portfolio with the highest concentration risk is considered compliant to the regulation of the Bank of Italy (Banca d'Italia, 2006). Both the total exposure of a portfolio and the number of loans are changed in order to analyse their impact on the indices of single-name concentration risk. The main results of this numerical application are that the Reciprocal of Hannah-Kay index with $\alpha = 3$, the Herfindahl-Hirschman index and the Hall-Tidemann index stress the impact of lager exposures. On the contrary, the Reciprocal of Hannah-Kay index with $\alpha = 0.5$ and the Theil index underline the importance of smaller exposures.

The paper is organized as follows. Section 2 defines the six properties of a single-name concentration index and the relationships among them. Sections 3, 4 and 5 investigate which properties are satisfied respectively by the Gini index, the industrial concentration indices and the Theil entropy index. In section 6 the five indices are used to measure the single-name concentration risk of six portfolios. Section 7 is devoted to conclusions.

2 Properties of a single-name concentration index

Consider a portfolio of n loans. The exposure of the loan i is represented by $x_i \geq 0$ and the total exposure of the portfolio is $\sum_{i=1}^{n} x_i = T$. In the following, a portfolio is denoted by the vector of the shares of the amounts of the loans $\mathbf{s} = (s_1, s_2, \ldots, s_n)$: the share $s_i \geq 0$ of i-th loan is defined as $s_i = x_i/T$. It follows that $\sum_{i=1}^{n} s_i = 1$. Whenever the shares of the portfolio $\mathbf{s}$ need to be ordered, the corresponding portfolio obtained by the increasing ranking of the shares will be denoted by $\mathbf{s}_{(.)} = (s_{(1)}, \ldots, s_{(n)})$. It is clear that any reasonable concentration measure C must satisfy $C(\mathbf{s}) = C(\mathbf{s}_{(.)})$. Whenever it is necessary, in order to remark the number n of the loans in the portfolio, the single-name concentration measure will be denoted with C_n.

The following six properties are desirable ones that a single-name concentration measure C should satisfy. Indeed they were born in a different framework, nevertheless their translation to credit analysis can be considered successful (cfr [3], [7] and [15]).

1. **(Transfer principle)** *The reduction of a loan exposure and an equal increase of a bigger loan that preserve the order must not decrease the concentration measure.* Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$ be two portfolios such that
$$s^*_{(k)} = \begin{cases} s_{(j)} - h & k = j \\ s_{(j+1)} + h & k = j+1 \\ s_{(k)} & \textit{otherwise}, \end{cases} \tag{1}$$
where
$$s_j < s_{j+1}, \qquad 0 < h < s_{(j+1)} - s_{(j)}, \qquad h < s_{(j+2)} - s_{(j+1)}. \tag{2}$$
Then $C(\mathbf{s}) \leq C(\mathbf{s}^*)$
2. **(Uniform distribution principle)** *The measure of concentration attains its minimum value, when all loans are of equal size.* Let $\mathbf{s} = (s_1, s_2, \cdots, s_n)$ be a portfolio of n loans. Then $C(\mathbf{s}) \geq C(\mathbf{s_e})$, where $\mathbf{s}_e$ is the portfolio with equal-size loans, that is $\mathbf{s_e} = (1/n, \ldots, 1/n)$.
3. **(Lorenz-criterion)** *If two portfolios, which are composed of the same number of loans, satisfy that the aggregate size of the k biggest loans of the first portfolio is greater or equal to the size of the k biggest loans in the second portfolio for* $1 \leq k \leq n$*, then the same inequality must hold between the measures of concentration in the two portfolios.* Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$ be two portfolios with n loans. If $\displaystyle\sum_{i=k}^{n} s^*_{(i)} \geq \sum_{i=k}^{n} s_{(i)}$ for all $k = 1, \ldots, n$, then $C(\mathbf{s}) \leq C(\mathbf{s}^*)$.
4. **(Superadditivity)** *If two or more loans are merged, the measure of concentration must not decrease.* Let $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n)$ be a portfolio of n loans, and $\mathbf{s}^* = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_m, s_{j+1}, \ldots, s_n)$ a portfolio of $n-1$ loans such that $s_m = s_i + s_j$. Then $C_n(\mathbf{s}) \leq C_{n-1}(\mathbf{s}^*)$.

5. **(Independence of loan quantity)** *Consider a portfolio consisting of loans of equal size. The measure of concentration must not increase with an increase in the number of loans.*
   Let $\mathbf{s}_{e,n} = (1/n, \dots, 1/n)$ and $\mathbf{s}_{e,m} = (1/m, \dots, 1/m)$ be two portfolios with equal-size loans and $n \geq m$, then $C_n(\mathbf{s}_{e,n}) \leq C_m(\mathbf{s}_{e,m})$.
6. **(Irrelevance of small exposures)** *Granting an additional loan of a relatively low amount must not increase the concentration measure. More formally, if s' denotes a share of a loan and a new loan with a share $\tilde{s} \leq s'$ is granted, then the concentration measure must not increase.*
   Let $\mathbf{s} = (s_1, s_2 \dots, s_n)$ be a portfolio of n loans with total exposure T. Then, there exists a share s' such that for all $\tilde{s} = \tilde{x}/(T+\tilde{x}) \leq s'$ the portfolio of $n+1$ loans $\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_{n+1}^*)$ with shares

$$s_i^* = \begin{cases} x_i/(T+\tilde{x}) & i = 1, 2, \dots, n \\ \tilde{x}/(T+\tilde{x}) & i = n+1 \end{cases}$$

   is considered. It holds that $C(\mathbf{s}) \geq C(\mathbf{s}^*)$.

A few remarks on the aforementioned properties can be useful. The first three properties have been proposed for the concentration of income distribution. In the first three properties the number n of loans of the portfolio is fixed, while in the others n changes. This means that the properties 4, 5 and 6 point out the influence of the number of the loans on the concentration measure. The principle of transfers and the Lorenz-criterion have been proposed at the beginning of the last century:the former has been introduced by Pigou in 1912 (see [16]) and Dalton in 1920 (see [5]), the latter is related to the Lorenz curve proposed in 1905 by Lorenz (see [14]). The property 4 can be applied more than one time by setting up the merge of three or more loans. Finally, the properties 4 and 5 have been suggested in the field of the industrial concentration where the issue of monopoly is very important.

Theorem 1 (Link among the properties).
If a concentration measure satisfies the properties 1 and 6, then it satisfies all the aforementioned six properties.

Proof. The outline of the proof is the following. It can be proved that a concentration index satisfying property 1 fulfills also properties 2 and 3. Further, if a concentration measure satisfies the properties 1 and 6, then it meets the property 4. Finally, properties 2 and 4 imply the property 5.

1. *Property 1 $\Rightarrow$ property 3*
   Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_n^*)$ be two portfolios of n loans each. Let the shares of the two portfolios be such that $\sum_{i=1}^k s_{(i)}^* \geq \sum_{i=1}^k s_{(i)}$
   $\forall k = 1, \dots, n-1$. Let $\boldsymbol{\Delta}$ denote the difference vector $\mathbf{s}_{(.)}^* - \mathbf{s}_{(.)}$, which is $\boldsymbol{\Delta} = (\Delta_1, \Delta_2, \dots, \Delta_n) = (s_{(1)}^* - s_{(1)}, s_{(2)}^* - s_{(2)}, \dots, s_{(n)}^* - s_{(n)})$. By construction $\sum_{i=1}^k \Delta_i \geq 0$ for $k = 1, \dots, n-1$ and $\sum_{i=1}^n \Delta_i = 0$. The idea is to construct, by induction, a sequence of t vector of shares $\mathbf{m}^1, \dots, \mathbf{m}^t$ such that

- $\mathbf{m}^1 = \mathbf{s}_{(.)}$ and $\mathbf{m}^t = \mathbf{s}^*_{(.)}$;
- $\mathbf{m}^h$ is obtained from $\mathbf{m}^{h-1}$ by a transfer of part of share of a certain loan to a bigger one.

Suppose that $\mathbf{m}^{h-1} = (m_1^{h-1}, \dots, m_n^{h-1})$ is already defined and satisfies the three restrictions:

1) $m_i^{h-1} \geq 0 \; i = 1, \dots, n$;
2) $\sum_{i=1}^n m_i^{h-1} = 1$;
3) the m_i^{h-1} are decreasingly ranked.

Introduce the vector $\boldsymbol{\Delta}^{h-1} = \mathbf{s}^*_{(.)} - \mathbf{m}^{h-1}$, with $\sum_{i=1}^n \Delta_i^{h-1} = 0$. Let α be the first value of the index i in $\{1, \dots, n\}$ such that $\Delta^{h-1} < 0$, which is

$$\alpha = \min_{i \in \{1,\dots,n\}} \{i \, : \, \Delta_i^{h-1} < 0\}.$$

By construction, $m_\alpha^{h-1} > s^*_{(\alpha)}$ and $m_\alpha^{h-1} = s^*_{(\alpha)} - \Delta_\alpha^{h-1}$. Consider now the loan β, where β is the first value of the index i in $\{1, \dots, \alpha\}$ that realizes the minimum of the strictly positive values of Δ_i^{h-1}

$$\beta = \min_{i \in \{1,\dots,\alpha\}} \{\Delta_i^{h-1} \, : \, \Delta_i^{h-1} > 0\}.$$

Such a value β always exists because the first non-null component of vector Δ^{h-1} cannot be negative. By construction, $m_\beta^{h-1} < s^*_{(\beta)}$ and $m_\beta^{h-1} = s^*_{(\beta)} - \Delta_\beta^{h-1}$. Since $\beta < \alpha$ and since the components of vector $\mathbf{m}^{h-1}$ are decreasingly ordered, it results $m_\beta^{h-1} \geq m_\alpha^{h-1}$. By a transfer of the positive quantity $\min(-\Delta_\alpha^{h-1}, \Delta_\beta^{h-1})$ from the loan α to the loan β, the vector $\tilde{\mathbf{m}}^h$ can be obtained with shares

$$\tilde{m}_i^h = \begin{cases} m_\beta^{h-1} + \min(-\Delta_\alpha^{h-1}, \Delta_\beta^{h-1}) & i = \beta \\ m_\alpha^{h-1} - \min(-\Delta_\alpha^{h-1}, \Delta_\beta^{h-1}) & i = \alpha \\ m_i^{h-1} & otherwise \end{cases}$$

The shares $\tilde{m}_i^h$ with $i = 1, 2, ..., n$ is decreasingly ordered and this vector is denoted by $\mathbf{m}^h$. The vector $\mathbf{m}^h$ is obtained from $\mathbf{m}^{h-1}$ by a transfer of a positive amount from the share α to a bigger share β. If the property 1 holds, then $C_n(\mathbf{m}^h) \geq C_n(\mathbf{m}^{h-1})$. Following this procedure it is possible to construct by induction a sequence of vector of shares: this sequence starts from the portfolio $\mathbf{s}$ and ends with the portfolio $\mathbf{s}^*$ by increasing the concentration at each step.

Finally, it should be showed that the number of iteration of the procedure is finite. At each step, a component of $\mathbf{m}^h$ is replaced by the respective component value of $\mathbf{s}^*$. The number of the components of $\mathbf{s}^*$ is finite, therefore the algorithm converges after a finite number of iterations. Finally, it holds that

$$C_n(\mathbf{s}) = C_n(\mathbf{m}^1) \leq \cdots \leq C_n(\mathbf{m}^{h-1}) \leq C_n(\mathbf{m}^h) \leq \dots C_n(\mathbf{m}^t) = C_n(\mathbf{s}^*)$$

2. *Property 1 ⇒ property 2*
   The proof is similar to the previous one. The idea is to construct a sequence of transfers which transforms any portfolio of n loans in the portfolio with equal-amount loans by decreasing the concentration at each step. Thus, the minimum value of the single-name concentration index is obtained when all the shares have the same value.
3. *Properties 1 and 6 ⇒ property 4*
   Let $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n)$ be a portfolio of n loans, and
   $\mathbf{s}^* = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_m, s_{j+1}, \ldots, s_n)$ a portfolio obtained by the merge $s_m = s_i + s_j$. If $s_i \leq s_j$ consider the portfolio
   $s^1 = (s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_{j-1}, s_m, s_{j+1}, \ldots, s_n)$ is obtained by a transfer from a smaller loan (s_i) to a bigger one (s_j). Since the property 1 holds, $C_n(\mathbf{s}) \leq C_n(\mathbf{s}^1)$.
   Now, if the property 6 is satisfied, the loan with null amount can be removed from $\mathbf{s}^1$ with no concentration increase. The result is the portfolio $\mathbf{s}^*$ and it holds that $C_n(\mathbf{s}) \leq C_n(\mathbf{s}^1) \leq C_{n-1}(\mathbf{s}^*)$, therefore the property 4 is true.
4. *Properties 2 and 4 ⇒ property 5*
   Since the property 2 is satisfied, $C_{n+1}(1/n, \ldots, 1/n, 0) \geq C_{n+1}(1/(n+1), \ldots, 1/(n+1))$. After a merge, by property 4, it follows that
   $C_n(1/n, \ldots, 1/n) \geq C_{n+1}(1/n, \ldots, 1/n, 0)$, and therefore
   $C_n(1/n, \ldots, 1/n) \geq C_{n+1}(1/(n+1), \ldots, 1/(n+1))$. This means that $C_n(\mathbf{s_{e,n}}) \geq C_{n+1}(\mathbf{s_{e,n+1}})$. By iteration, the property 5 holds true.

3 An inequality index: Gini coefficient

The Gini coefficient (G) was introduced by Gini at the beginning of the last century [8]. It is a measure of the “distance” between the equalitarian and the considered situation. In the case of a portfolio of n loans $\mathbf{s} = (s_1, \ldots, s_n)$ it is defined as

$$G = \frac{n+1}{n-1} - \frac{2}{n-1} \sum_{i=1}^{n} (n-i+1) s_{(i)}. \tag{3}$$

It is a normalized index, since its range is the interval $[0, 1]$: it assumes value 0 if the loans have the same amount, while it equals 1 if the total amount corresponds to an unique loan.

The major drawback is that the G index does not take into account the portfolio size, therefore it can be considered more an inequality index than a concentration index.

Theorem 2. *The G index satisfies the properties 1, 2, 3 and 5.*

Proof. Properties 1, 2 and 3
Let $\mathbf{s} = (s_1, \ldots, s_n)$ a portfolio to which the transfer (2) is applied and so the portfolio $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ is obtained.

The difference between the G index of these two portfolios is

$$\begin{aligned} G(\mathbf{s}^*) - G(\mathbf{s}) &= \frac{2}{n-1}\left[(n-j+1)\left(s_{(j)} - s^*_{(j)}\right) + (n-j)\left(s_{(j+1)} - s^*_{(j+1)}\right)\right] \\ &= \frac{2}{n-1}\left[(n-j+1)h + (n-j)(-h)\right] \\ &= \frac{2h}{n-1} > 0. \end{aligned}$$

Then $G(\mathbf{s}^*) > G(\mathbf{s})$.

As the Theorem 1 states, the property 1 implies the properties 2 and 3, which are therefore satisfied by the G index.

Property 5

Consider two portfolios with equal-amount loans:

$$\mathbf{s}_{e,n} = (1/n, \ldots, 1/n) \text{ and } \mathbf{s}_{e,m} = (1/m, \ldots, 1/m).$$

From the definition of the G index (3), it follows that $G(\mathbf{s}_{e,n}) = G(\mathbf{s}_{e,m}) = 0$. Hence, it holds that $G_n(\mathbf{s}_{e,n}) \leq G_m(\mathbf{s}_{e,m})$.

4 Industrial concentration indices

4.1 Hannah-Kay index

For the industrial concentration Hannah and Kay [10] have proposed the following index (HK)

$$HK = \left(\sum_{i=1}^{n} s_i^{\alpha}\right)^{\frac{1}{1-\alpha}} \quad \text{with } \alpha > 0 \text{ and } \alpha \neq 1.$$

The HK index is inversely proportional to the level of concentration: by increasing concentration the HK index decreases. For this reason in this paper, as in Becker, Dullmann and Pisarek [3], the Reciprocal of Hannah-Kay (RHK) index is considered:

$$RHK = \left(\sum_{i=1}^{n} s_i^{\alpha}\right)^{\frac{1}{\alpha-1}} \quad \alpha > 0 \text{ and } \alpha \neq 1, \tag{4}$$

so that the RHK index is proportional to the level of concentration.

For a portfolio with equal-size loans the RHK index is

$$RHK = \left[\sum_{i=1}^{n} \left(\frac{1}{n}\right)^{\alpha}\right]^{\frac{1}{\alpha-1}} = \frac{1}{n}.$$

If the portfolio consists of only one non-null share, the RHK index is equal to 1. The role of the elasticity parameter α is to decide how much weight to attach to the upper portion of the distribution relative to the lower. High α gives greater weight to the role of the highest credit exposures in the distribution and low α emphasizes the presence or the absence of the small exposures.

Theorem 3. *The RHK index satisfies all the six properties considered in Section 2.*

Proof. From theorem 1 if the 1 and 6 properties are satisfied, all the six properties of a concentration measure are satisfied.
Property 1 [1]
Let $\mathbf{s}$ and $\mathbf{s}^*$ two portfolios that satisfy the condition (2). The following difference is computed

$$f(h) = RHK(\mathbf{s}^*) - RHK(\mathbf{s}) = \left(\sum_{k \neq i,j} s_k^\alpha + (s_j + h)^\alpha + (s_j - h)^\alpha \right)^{\frac{1}{\alpha-1}} - \left(\sum s_k^\alpha \right)^{\frac{1}{\alpha-1}} .$$

The function $f(h)$ is continuous for $h > 0$ and $\lim_{h \to 0} f(h) = 0$. The derivative of $f(h)$ is

$$\frac{\partial f(h)}{\partial h} = \frac{\alpha}{\alpha - 1} \left(\sum_{k \neq i,j} s_k^\alpha + (s_j + h)^\alpha + (s_j - h)^\alpha \right)^{\frac{2-\alpha}{\alpha-1}} \left[(s_j + h)^{\alpha-1} - (s_i - h)^{\alpha-1} \right] . \tag{5}$$

In order to determine the sign of this derivative, two cases are considered

1. $0 < \alpha < 1$
   In the equation (5) the first and the third factors of the product are negative and the second factor is positive, hence the derivative is positive.
2. $\alpha \geq 1$
   In the equation (5) all the factors are positive, hence the derivative is positive.

Property 6
Let $\mathbf{s}$ and $\mathbf{s}^*$ two portfolios that satisfy the conditions given in the property 6. The following difference is computed

$$g(\tilde{x}) = RHK(\mathbf{s}^*) - RHK(\mathbf{s}) = \left[\sum_{i=1}^{n} \left(\frac{x_i}{T + \tilde{x}} \right)^\alpha + \left(\frac{\tilde{x}}{T + \tilde{x}} \right)^\alpha \right]^{\frac{1}{\alpha-1}} - \left[\sum_{i=1}^{n} s_i^\alpha \right]^{\frac{1}{\alpha-1}} .$$

The function $g(\tilde{x})$ is continuous for $\tilde{x} > 0$ and $\lim_{\tilde{x} \to 0} g(\tilde{x}) = 0$, so the entry of a new loan with insignificant exposure $\tilde{x}$ in the portfolio has insignificant impact on the RHK index.

[1] The proof is similar to the one suggested by Becker, Dullmann and Pisarek [3].

The derivative of $g(\tilde{x})$ is computed by obtaining

$$\frac{\partial g(\tilde{x})}{\partial \tilde{x}} = \frac{\alpha}{\alpha-1}\left[\sum_{i=1}^{n}\left(\frac{x_i}{T+\tilde{x}}\right)^{\alpha} + \frac{\tilde{x}}{T+\tilde{x}}^{\alpha}\right]^{\frac{2-\alpha}{\alpha-1}} \frac{\sum_{i=1}^{n} x_i\left(\tilde{x}^{\alpha-1} - x_i^{\alpha-1}\right)}{(T+\tilde{x})^{\alpha+1}}. \quad (6)$$

In order to determine the sign of this derivative for $\tilde{s} < \tilde{s}'$, two cases are considered:

1. $0 < \alpha < 1$
   In the equation (6) the first factor of the product is negative and the second and the third factors are positive, hence the derivative is negative.
2. $\alpha \geq 1$
   In the equation (5) the first and the second factors are positive and the third factor is negative, hence the derivative is negative.

This means that even if the introduction of a new loan with exposure $\tilde{x}$ causes a negligible change in the RHK index, it slightly decreases.
Set the equation (6) equal to zero, the superior limit s' for $\tilde{s}$ is obtained

$$s' = \left[\sum_{i=1}^{n} s_i^{\alpha}\right]^{\frac{1}{\alpha-1}} = RHK.$$

It follows that if a new loan has a share $\tilde{s}$ higher than the RHK index, the effect of the new loan in reducing the share of the existing large exposures is offset to some extent by the fact that its exposure is large.

The next index represents a particular case of the RHK index for a given value of the elasticity parameter α.

4.1.1 Herfindahl-Hirschman index

By considering $\alpha = 2$, the RHK index (4) becomes

$$HH = \sum_{i=1}^{n} s_i^2$$

the Herfindahl-Hirschman index (HH) proposed by Herfindahl [11] as an industrial concentration index, whose root has been proposed by Hirschman [12]. For this reason, this index is known as Herfindahl-Hirschman index. It is defined as the sum of squared portfolio shares of all borrowers.

By considering the square of the portfolio share s_i in the HH index, small exposures affect the level of concentration less than a proportional relationship.
The main advantage of the HH index is that it satisfies all the six properties of an index of credit concentration, because it is a particular case of the RHK index.

4.2 Hall-Tidemann index

The last industrial concentration index here analysed has been proposed by Hall and Tidemann (HT) and defined as

$$HT = \frac{1}{2\sum_{i=1}^{n}(n-i+1)s_{(i)} - 1}.$$

This index weights each loan with a value depending on its rank: this feature gives more importance to big loans and to the total number of the loans. If the amounts of all the loans are equal, then it results

$$HT = \frac{1}{2n^{-1}\sum_{i=0}^{n}(n-i+1) - 1} = \frac{1}{n}.$$

If there is only one loan in the portfolio, the value of the index is one. An important result in the literature is the link between the HT and the G indices

$$HT = \frac{1}{n-(n-1)G}.$$

Theorem 4. *The HT satisfies all the six properties considered in section 2.*

Proof. Property 1
Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$ be two portfolios of n loans each one, as in property 1. Then, since the Gini coefficient G satisfies the property 1, it holds $G(\mathbf{s}) \leq G(\mathbf{s}^*)$. Hence

$$\begin{aligned} G(\mathbf{s}) &\leq G(\mathbf{s}^*) \\ n-(n-1)G(\mathbf{s}) &\geq n-(n-1)G(\mathbf{s}^*) \\ HT(\mathbf{s}) &\leq HT(\mathbf{s}^*). \end{aligned}$$

Property 6
Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_n^*, s_{n+1}^*)$ be two portfolios of n and $n+1$ loans, respectively, as stated in property 6. Consider the difference $HT(\mathbf{s}) - HT(\mathbf{s}^*)$:

$$
\begin{aligned}
&HT(\mathbf{s}) - HT(\mathbf{s}^*) = \\
&= \frac{1}{2[ns_{(1)} + (n-1)s_{(2)} + \cdots + s_{(n)}] - 1} - \frac{1}{2[(n+1)\tilde{s} + ns_{(1)} + (n-1)s_{(2)} + \cdots + s_{(n)}] - 1} \\
&= \frac{T}{2[nx_{(1)} + (n-1)x_{(2)} + \cdots + x_{(n)}] - T} - \frac{T + \tilde{x}}{2[nx_{(1)} + (n-1)x_{(2)} + \cdots + x_{(n)}] - T + \tilde{x}(2n+1)} \\
&= c \cdot [T(2n+1) - 2[nx_{(1)} + (n-1)x_{(2)} + \cdots + x_{(n)}] + T] \\
&= c \cdot [2nT + 2T - 2[nx_{(1)} + (n-1)x_{(2)} + \cdots + x_{(n)}]] \\
&> c \cdot [2nT + 2T - 2nT] > 0.
\end{aligned}
$$

where c is a constant greater than 0. It follows that $HT(\mathbf{s}) > HT(\mathbf{s}^*)$ and therefore property 4 holds true.

5 An index from information theory: Theil entropy index

The RHK index is undefined for $\alpha = 1$ but its behaviour can be analysed when α is close to 1. Let $\alpha = 1 + h$, the limit of the RHK for $h \to 0$ is computed. By applying the Taylor expansion, for $h \to 0$ we obtain

$$\sum_{i=1}^{n} s_i^{h+1} \simeq \sum_{i=1}^{n} (s_i + hs_i \log s_i) = 1 + h \sum_{i=1}^{n} s_i \log s_i. \tag{7}$$

By computing the logarithm of the RHK index and by considering the result (7), it is obtained [2]

$$\lim_{\alpha \to 1} \log RHK = \lim_{h \to 0} \log \left(\sum_{i=1}^{n} s_i^{h+1} \right)^{\frac{1}{h}} = \lim_{h \to 0} \left(\frac{1}{h} h \sum_{i=1}^{n} s_i \log s_i \right) = \sum_{i=1}^{n} s_i \log s_i. \tag{8}$$

By considering the Theil (TH) entropy index [17]

$$TH = \sum_{i=1}^{n} s_i \log \frac{1}{s_i}, \tag{9}$$

the result (8) can be written as a transformation of the TH index, so obtaining the following relationship

$$\lim_{\alpha \to 1} RHK = exp(-TH).$$

Analogously to the HK index, the TH index is inversely proportional to the level of concentration. To obtain a directly proportional measure of concentration from

[2] if for any i, $s_i = 0$, then the quantity $s_i \log s_i$ is conventionally defined as 0

the expression (9), it is preferred to consider

$$DTH = max\{TH\} - TH = \log n - TH = \sum_{i=1}^{n} s_i \log s_i + \log n \tag{10}$$

When all the loans have the same exposure, the DTH index is

$$DTH = \log n - \log n = 0.$$

If a portfolio with only one non-null share is considered, the DTH index is equal to $\log n$.

By considering the logarithm of the portfolio share s_i, DTH gives relatively more weight to smaller loans.

Theorem 5. *The DTH index satisfies the properties 1,2,3,4, and 5.*

Proof. Property 1
Consider the difference between the TH indices of the two portfolios $\mathbf{s}$ and $\mathbf{s}^*$ that satisfy the assumptions of the property 1:

$$DTH(\mathbf{s}^*) - DTH(\mathbf{s}^) = (s_j + h)\log(s_j + h) + (s_i - h)\log(s_i - h) - s_j \log s_j - s_i \log s_i. \tag{11}$$

Because the second derivative of the function $x \log x$ is non-negative on the interval $(0,1)$, $x \log x$ is a convex function. Let

$$s_j = \alpha(s_j + h) + (1 - \alpha)(s_i - h)$$

$$s_i = (1 - \alpha)(s_j + h) + \alpha(s_i - h)$$

where $h > 0$, $s_i < s_j$ and $\alpha = \dfrac{s_j - s_i + h}{s_j - s_i + 2h} \in (0,1)$.

Because $x \log x$ is a convex function, the following inequalities are satisfied

$$s_j \log s_j \leq \alpha(s_j + h)\log(s_j + h) + (1 - \alpha)(s_i - h)\log(s_i - h))$$

$$s_i \log s_i \leq (1 - \alpha)(s_j + h)\log(s_j + h) + \alpha(s_i - h)\log(s_i - h)).$$

It follows that

$$s_i \log s_i + s_j \log s_j \leq (s_j + h)\log(s_j + h) + (s_i - h)\log(s_i - h)).$$

This means that the difference (11) is positive and so the property 1 is satisfied. *Property 4*
Consider two portfolios of loans $\mathbf{s}$ and $\mathbf{s}^*$ as in property 4. Then:

$$
\begin{aligned}
DTH_{n-1}(\mathbf{s}^*) - DTH_n(\mathbf{s}) &= s_m \log s_m + \log(n+1) - [s_i \log s_i + s_j \log s_j + \log n] \\
&= s_m \log s_m - s_i \log s_i - s_j \log s_j + \log(n+1) - \log n \\
&= \log \frac{s_m^{s_m}}{s_i^{s_i} s_j^{s_j}} + \log \frac{n+1}{n} \\
&= \log \frac{(s_i + s_j)^{s_i + s_j}}{s_i^{s_i} s_j^{s_j}} + \log \frac{n+1}{n} \\
&= \log \left(\frac{s_i + s_j}{s_i} \right)^{s_i} \left(\frac{s_i + s_j}{s_j} \right)^{s_j} + \log \frac{n+1}{n} > 0, \qquad (12)
\end{aligned}
$$

since all the arguments of the logarithms in (12) are greater than 1.

6 Numerical applications

In this section six portfolios of loans are considered and the indices presented in the previous sections are calculated on them.
For the construction of the most concentrated portfolio, the large exposure limits of the Bank of Italy (Banca d'Italia, 2006) is considered. In this analysis the exposure of the portfolio T is 1,000 euros. Therefore, the minimum regulatory capital charge of 8% is 80 euros and this is considered the capital requirement of the bank. The Bank of Italy establishes that an exposure is defined as *large* if it amounts to 10% or more of the bank's regulatory capital, in this case an exposure is large if it is greater than or equal to 8 euros. According to the Bank of Italy's regulation, a large exposure must not exceed 25% of the regulatory capital, in this case 20 euros. The sum of all large exposures is limited to eight times the regulatory capital, which corresponds to 640 euros in this case. By considering this regulation, the portfolio with the highest concentration risk P1 consists of 32 exposures equal to 20 euros, 51 equal to 7 euros and one equal to 3 euros. Hence, the total exposure of the portfolio P1 is 1,000 euros and its number of loans is 84.

In order to obtain the portfolio P2, each exposure of 20 euros in the portfolio P2 is divided into two exposures of 10 euros. It follows that the total exposure of the portfolio P2 remains constant ($T = 1,000$) and the number of the loans of the portfolio increases ($n = 116$). Moreover, the portfolio P3 is obtained from the portfolio P2 by merging two exposures of 10 euros in one of 20 euros. From the portfolio P3, by neglecting the exposure of 20 euros the portfolio P4 is defined. Finally, the last two portfolios are obtained by introducing in P4 a medium exposure of 7 euros for the portfolio P5 and a low exposure of 3 euros for the portfolio P6. It is important to highlight that both the total exposure T and the number of loans n can change in these six portfolios.

$$\underbrace{20\ldots20}_{32}\ \underbrace{7\ldots7}_{51}\ 3 \qquad T=1,000 \qquad n=84 \qquad \mathbf{P1}$$
$$\underbrace{10\ldots10}_{64}\ \underbrace{7\ldots7}_{51}\ 3 \qquad T=1,000 \qquad n=116 \qquad \mathbf{P2}$$
$$20\ \underbrace{10\ldots10}_{62}\ \underbrace{7\ldots7}_{51}\ 3 \qquad T=1,000 \qquad n=115 \qquad \mathbf{P3}$$
$$\underbrace{10\ldots10}_{62}\ \underbrace{7\ldots7}_{51}\ 3 \qquad T=980 \qquad n=114 \qquad \mathbf{P4}$$
$$\underbrace{10\ldots10}_{62}\ \underbrace{7\ldots7}_{52}\ 3 \qquad T=987 \qquad n=115 \qquad \mathbf{P5}$$
$$\underbrace{10\ldots10}_{62}\ \underbrace{7\ldots7}_{51}\ 3\,3 \qquad T=983 \qquad n=115 \qquad \mathbf{P6}$$

Table 1 summarizes the values of the single-name concentration indices considered for the six portfolios. The portfolio P3 shows a higher single-name concentration of the portfolios P4 and P5, this ordering is satisfied by all the indices except the RHK index with $\alpha = 0.5$. This result is mainly due to the characteristic of the RHK with $\alpha = 0.5$ to stress the importance of smaller exposures.

	G	**HH**	**HT**	**DTH**	**RHK**$(\alpha = 3)$	**RHK**$(\alpha = 0.5)$
P1	264.63	15.31	16.12	138.32	16.54	12.77
P2	90.82	8.91	9.47	17.47	9.03	8.69
P3	100.21	9.11	9.65	22.68	9.35	8.79
P4	91.42	9.07	9.65	17.61	9.19	8.85
P5	91.66	8.99	9.56	17.65	9.11	8.77
P6	105.85	8.84	9.71	40.21	9.15	8.78

Table 1 The values of the concentration indices are computed for the six portfolios and multiplied by 1000.

It is interesting the comparison of the portfolios P4 and P6, even if it falls in the property 6. For the RHK index with $\alpha = 0.5$, $\alpha = 3$ and the HH index, the results in Table 1 are coherent with the proof in Section 4 that these indices satisfy the property 6 of the irrelevance of small exposures. Moreover, the results in Table 1 show that the G and the DTH do not satisfied the property 6, according to what showed in the previous section. Finally, for the HT index the result can be incoherent with the proof in the Subsection 4.2. In particular, the HT index satisfies the property 6 but its superior limit s' of small exposures is lower than 3 (the exposure $\widetilde{s}$ added to the portfolio). Indeed, by adding an exposure equal to one to the portfolio, the HT index decreases.

The comparison between the portfolios P3 and P2 falls in the subadditivity property 4. Even if the G index does not satisfy this property, in this particular case G satisfies the ordering established by the property 4. The portfolio P5 is obtained by adding a medium exposure to the portfolio P4, for this reason the ordering of the concentration risks of these portfolios is ambiguous. From the portfolio P4 to P5, the G and the DTH indices show an increase of the concentration risk, on the contrary the other indices a decrease.

All the indices agree that the concentration risk decreases from P4 to P2. It follows that the impact of the increase of the number of loans in the portfolio is higher than the impact of larger loans. Finally, the portfolio P5 shows a higher concentration risk than that of the portfolio P6. It is important to highlight that only the HH index satisfies this ordering.

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