Liar's Dice and Binomial Random Variables

Introduction

Probability is one of the most important areas of study in Mathematics as it governs almost every decision that is made in the world today. From weather prediction, to predicting stock prices and winners of sports competitions, probability comes up in every area where prediction of any kind is involved. Therefore, it is imperative that students begin to see many of the ways that we can express probabilities and group events into certain types or "distributions". One such distribution is the Binomial distribution.

Aim of the Workshop

The aim of this workshop is to introduce binomial random variables in the context of a game of chance. Liar's Dice (or Perudo) is an ancient game where having an understanding about the probability of certain combinations of dice rolls is the difference between winning and losing. We introduce the binomial formula and demonstrate how we can use it to work out probabilities that would be otherwise complicated and cumbersome to calculate.

Learning Outcomes

By the end of this workshop students should be able to:

- Explain, in their own words, what each term of the binomial formula represents
- Be able to apply this formula to examples involving the rolling of dice
- Describe the application of the theory to the game Liar's Dice.

Materials and Resources

Each student will require: at least 4 six–sided dice (Max. 6), a cup or other non–transparent container (or they can use their hands to cover their dice), activity sheets, paper, pens

Key Words

Distribution
describes the shape of the data when plotted on a histogram.

Bernoulli trial
Experiment where there are only two possible outcomes, usually with one defined as "success" and one defined as "failure" and where the probability of success is the same every time we perform the experiment.

Binomial Random Variable
is the number of successes (usually labelled k) in n independent Bernoulli trials.

Independent events
the occurrence of one event does not affect the probability of any of the others.

Binomial forumula

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Liar's Dice and Binomial Random Variables: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:10)	**Introduction of Bernoulli trials and Binomial Formula**	– Introduction to what a Bernoulli trial is and how we recognise it: Two outcomes: success or failure Examples: win or lose a game etc. – Extend the idea of one Bernoulli trial to many **independent trials** and introduce the binomial formula. (Remind the students of the meaning of **independent events** in probability with examples). – Introduce the Binomial formula and explain what each part represents (this may also be a useful revision exercise). – Discuss the other uses for the formula in fields such as medicine, weather and sports.
15 mins (00:25)	**Activity Sheet 1**	– Students attempt the questions on the **Activity Sheet**. – (They may need guidance in interpreting the various parts of the formula and filling in the correct values.)
10 mins (00:35)	**Explaining the rules of Liar's Dice**	– Discuss the rule sheet for the game (See **Appendix – Note 1**) and clarify these by going through some examples to demonstrate calling a bluff or calling a spot–on bet work in the game (See **Appendix – Note 2**). (You may wish to use one of the examples in the activity sheet to emphasise how likely/unlikely certain bids are to have occurred.)
25 mins (01:00)	**Let the students play the game**	– Students play the game. – After playing for 5–10 minutes, pause the games and ask students to try **Activity Sheet 3**. – This newly found knowledge from Activity Sheet 3 should provide students with opportunity to better strategize on bets or bluffs during the next game (**see Note 3**).

Liar’s Dice Appendix

Note 1: Liars Dice – How to play

- Each player starts with between 4 and 6 dice.
- To decide who goes first:

  Each player rolls one dice and the player that rolls highest will start the game. In the event of a tie, re–roll the dice until a clear winner is found.
- At the start of each round all players roll all of their dice inside of their dice cups and place them covered on the table.
- After taking time to look at their dice one player starts by placing a bet of a certain number of dice of a chosen face value (1, 2, 3, 4, 5, 6) being on the table (i.e. under ALL the cups)
- Each player then takes turns increasing the bet until one player believes the person before them has placed and incorrect bet (a **bluff**) or has predicted an exact number of dice on the table (a **spot on** bet) (see the **rules** section for more details).
- Once one player calls out the previous player’s bet, every player reveals their dice and the correct number of dice matching the current bet are counted. Depending on the call made and the number of dice matching the bet of the bluff or spot on bet, one player (or more) will lose a dice depending on the result (see **rules** below).
- After the result of the call is displayed by all players, all players roll their dice again and the next round begins with the player to the left of the player that started the previous round (counter–clockwise).
- The game continues until only one player has dice remaining and is declared the winner.

Placing bets

On their turn the player can do one of the following:

- Bet on the same number of dice of a HIGHER face value than the previous bet **(a player can never lower the face value only)**
- Bet on a higher number of dice of the same face value than the previous bet
- Bet on a higher number of dice AND any face value (higher OR lower)

Calling on the previous bet

Instead of placing a bet a player can choose to challenge the previous bet as either a bluff (usually with the call of “bluff”) or claim the previous bet was “spot on” (i.e. there is exactly the number of dice they bet on the table). The outcomes are as follows:

Call of “liar”

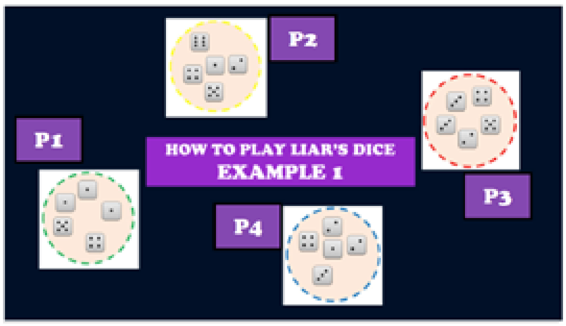
- If there are fewer than the bet number of dice on the table, then the player who placed the bet loses a dice
- If there are equal or more dice than the placed bet on the table, then the player who called “bluff” loses a dice

Call of “spot on” bet

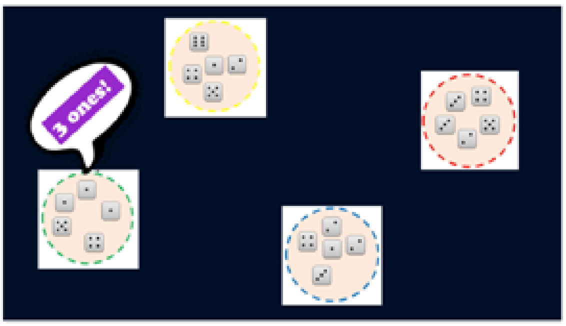
- If the bet matches the number of dice on the table, then every player other than the player who called “spot on” loses a dice
- If the bet does not match the number of dice on the table, then the player who called “spot on” loses a dice.

Note 2: Liar's Dice – How to play Examples

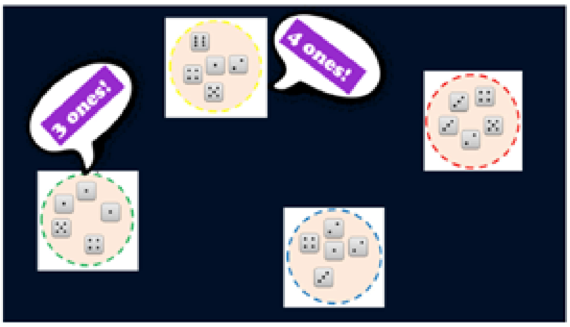
Example 1: The following is an example of how calling bluff on a bet works:



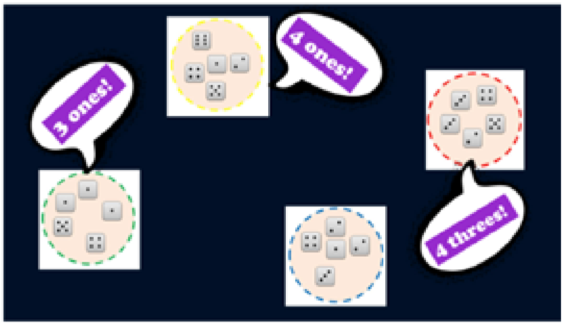
All students roll their dice and hide them under their cups. Each student checks their dice and one takes their turn placing a bet.



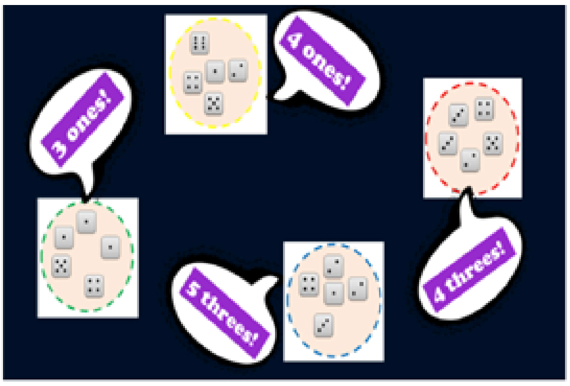
Player 1 rolled three ones, so they are certain that there are at least 3 ones on the table so they place their bet.



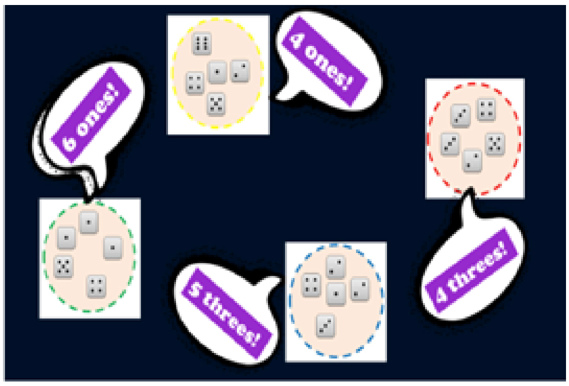
Player 2 increases the bet of the number of ones on the table from 3 to 4.



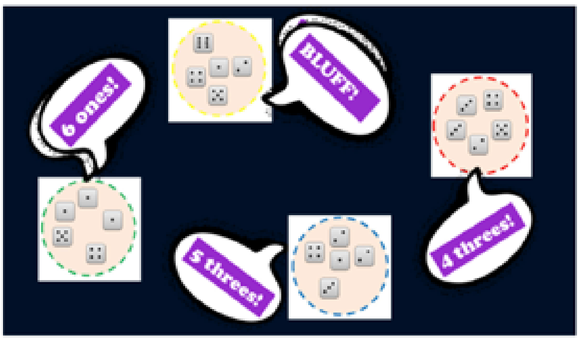
Player 3 decides to increase the face value of the dice from one to three since they have rolled 2 threes but 0 ones.

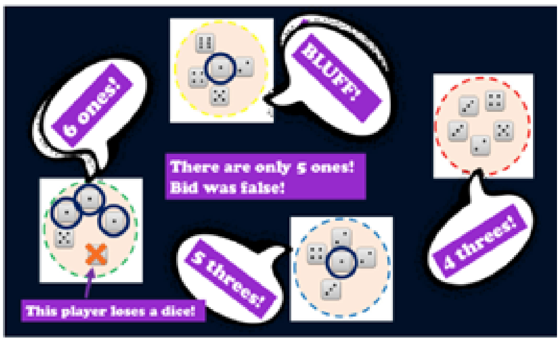


Player 4 increases the bid to 5 threes. Despite only rolling 1 three, they are hoping that there are at least 4 more on the table.



Player 1 increases the bid to 6 ones. (Note that they could not say 5 ones as you cannot decrease the face value only.)

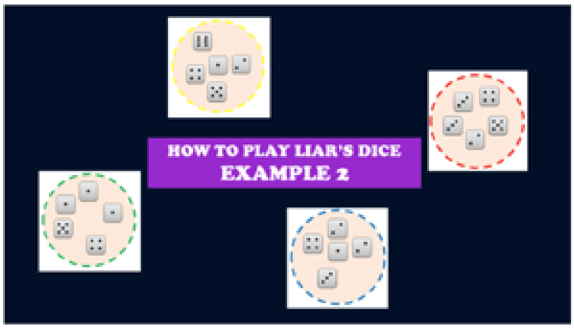


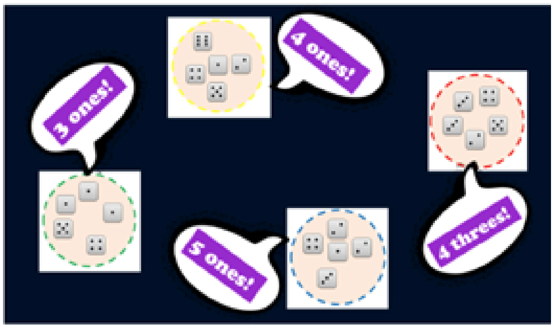


Player 2 calls bluff, believing that the previous bid of "6 ones" is false. (They think there are less than 6 ones on the table.)

All players reveal their dice and the number of ones is counted. There are 5 ones so the bid was a bluff, so player 1 loses a dice! This dice is removed from the game and cannot be used again.

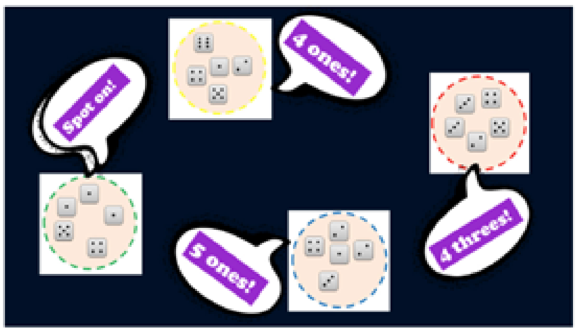
Example 2: The following is an example of calling a spot on bid





All students roll their dice and hide them under their cups. Each student checks their dice and one takes their turn placing a bet.

All bids progress as in example 1, however player 4 now bids that there are 5 ones on the table.



Player 1 calls the previous bid spot on, claiming that there are exactly 5 ones on the table

Since there are 5 ones on the table the spot on bid was correct. This means that every other player loses a dice. These dice are removed from the game and cannot be used again.

Note 3: Answers to Activity 3 – what the teacher sees

VARIABLE	SYMBOL	VALUE FOR TEACHER'S PROBLEM
Number of trials	n	**15**
Number of successes wanted	k	**5**
Probability of Success on one trial	p	**1/6**
Probability of Failure on one trial	$q = 1 - p$	**5/6**

P (Paul is correct) = *P* (**5** *ones in* **15** *dice*) =

$$\binom{15}{5}\left(\frac{1}{6}\right)^{5}\left(\frac{5}{6}\right)^{15-5} = \ldots\ [\textit{calculator}]\ \ldots \approx 0.06$$

The Binomial Formula – Activity 1

In all of the following examples we make use of the Binomial Random Variable formula for n **independent** random trials

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where

X = event of interest (e.g. getting heads on a coin toss)
n = total number of trials or experiments
k = number of desired successes { k = 0, 1, 2, ..., n}
p = probability of success
1 – p = probability of "failure"

A fair coin is thrown 10 times.
We count the number of heads we get in the 10 coin tosses. Can you find the following probabilities?

(a) the probability that you get heads 4 times

n =
k =
p =
1 – p =

Hence using the formula:

$P(\#heads = 4) =$

Hint: In probability we can use the following rule for independent Binomial experiments. If we have 7 trials, say, then:

$$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7)$$

and this is true for any number of events

The Binomial Formula – Activity 1

(b) the probability of 8 or more heads

n =
k =
p =
1 – p =
Hence using the formula:

$$P(\#heads \geq 8) =$$

Hint: In probability we can use the following rule for independent Binomial experiments. If we have 7 trials, say, then:

$$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7)$$

and this is true for any number of events

The Binomial Formula – Activity 2

6 **fair** six–sided dice are rolled. Can you work out the following probabilities?

NOTE: a **fair** dice is one where all numbers have EQUAL probability of being rolled

(a) Find the probability that you roll two 1s.

n =
k =
p =
1 – p =
Hence using the formula:

$$P(\#ones = 2\ ) =$$

(b) Find the probability that you roll all ones

n =
k =
p =
1 – p =
Hence using the formula

$$P(\#ones = 6\ ) =$$

The Binomial Formula – Activity 2

Here are some more questions if you've gotten this far.
Remember the steps we have in all the other questions.

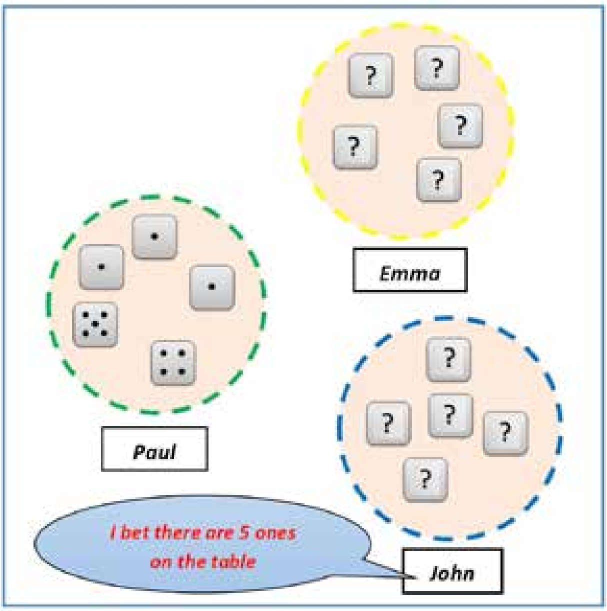
6 **fair** six–sided dice are rolled. Can you work out the following probabilities?

(c) Find the probability that you roll more than 4 ones

(d) Find the probability that you roll no ones

The Binomial Formula – Activity 3

Paul, Emma and John are playing Liar's Dice. Each has 5 dice and Paul rolls 3 ones, 1 four and 1 five. After taking turns to place bids John claims there are "5 ones on the table".



Paul then claims that John's bid is **"spot on"** (Paul thinks there are **exactly** 5 ones on the table).

Remember: Paul can only see **his** dice, he does **not** know what Emma and John have rolled!

Q: what is the probability that Paul's claim of "spot on" is correct?

Start by filling in the blanks in the following sentence

Paul has rolled _____ ones, meaning that there must be _____ ones in the remaining _____ dice on the table for there to be exactly 5 ones.

We can use the binomial formula to work out a probability for the above statement but we first need to know a few things:

VARIABLE	SYMBOL	VALUE FOR PAUL'S PROBLEM
Number of trials	n	
Number of successes wanted	k	
Probability of Success on one trial	p	
Probability of Failure on one trial	$q = 1 - p$	

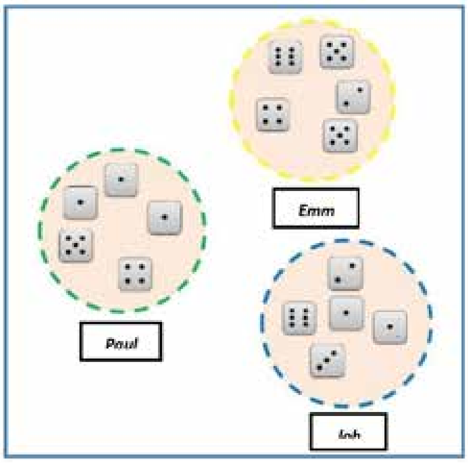
The probability that Paul's bet is exactly "spot on" is therefore:

$$P(\text{Paul is correct}) = P(\_\_\text{ones in }\_\_\text{dice}) =$$

Would you recommend Paul to stick with his bet? Why?

The Binomial Formula – Activity 3

Everyone reveals their dice and the results are shown in the diagram below.



Was Paul correct?

Who loses dice?

Pretend that you are Paul and that the teacher of the class comes over just before you all reveal your dice. Since the teacher cannot see any of the dice on the table, what would their answer be for the probability of your claim of "**spot on**" being correct?

Hint: Which variables in the table are different this time?

VARIABLE	SYMBOL	VALUE FOR TEACHER'S PROBLEM
Number of trials	n	
Number of successes wanted	k	
Probability of Success on one trial	p	
Probability of Failure on one trial	$q = 1 - p$	

$$P(\text{Paul is correct}) = P(\_\_\text{ones in }\_\_\text{dice}) =$$