Counting in Two Ways

Prof. Mark Flanagan
School of Electrical and Electronic Engineering
University College Dublin

Example Problems:

Example 1. 17 contestants took part in a mathematics contest with 9 problems. Each problem was solved by exactly 11 contestants. Prove that there exists a pair of contestants who, between them, solved all 9 problems.

Example 2 (IMO 1998 Q2). In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "Pass" or "Fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants.

Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b} \,.$$

Homework Problems:

Problem 1. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Problem 2 (Generalization of Example 1). Suppose that c contestants took part in a mathematics contest with p problems. Each problem was solved by at least k contestants. Prove that there exists a pair of contestants who, between them, solved all p problems provided that

$$p < \frac{c(c-1)}{d(d-1)} \, ,$$

where $d = c - k$.

Problem 3. Suppose that 15 contestants took part in a mathematics contest with 15 problems. Each contestant solved exactly 6 problems. Show that there exists a pair of contestants who solved at least 3 problems in common.

Problem 4. 8 singers participate in an art festival where m songs are performed. Each song is performed by 4 singers, and each pair of singers performs together in the same number of songs. Determine the minimum value of m.

Problem 5. A group of 10 people went into a bookshop. It is known that

(1) Everyone bought exactly 3 books;

(2) For every two persons, there is at least one book that both of them bought.

What is the least number of people that could have bought the book purchased by the greatest number of people?

[**NOTE:** On the next page, I have provided hints for the problems which you can look at if you get stuck. But I suggest to try each problem for a while first before looking at the corresponding hint.]

Hints for Homework Problems:

Problem 1. Create an incidence matrix, with rows corresponding to problems, and columns corresponding to contestants. Then count, in two different ways, the number of pairs of 1s that occur in the same row.

Problem 2 (Generalization of Example 1). You can use either the method of Solution 1 from the lecture or that of Solution 2 from the lecture; both can be made to work. The method of Solution 1 might involve less algebra.

Problem 3. Create an incidence matrix, with rows corresponding to problems, and columns corresponding to contestants. Then count, in two different ways, the number of pairs of 1s that occur in the same row. You may need to consider the total number of 1s in the matrix and how these 1s might be divided among the rows.

Problem 4. Create an incidence matrix, with rows corresponding to singers, and columns corresponding to songs. Now count, in two different ways, the number of pairs of 1s that occur in the same column. Next, consider divisibility properties of this number.

Problem 5. Create an incidence matrix, with rows corresponding to people, and columns corresponding to books. Notice that while there are 10 rows, the number of columns is unknown (we can call this b, but note that reasoning involving b may not be very helpful as it may possibly be very large). Let k denote the maximum number of 1s in any column of the matrix. What is the question asking us to prove about the value of k? Now, by counting (in two different ways) the number of pairs of 1s that occur in the same column, it should be possible to prove that $k \geq 4$. However, there is a catch: it can be checked that $k = 4$ is actually not possible to achieve (why?). Does a configuration (incidence matrix) exist that achieves $k = 5$?

EXAMPLE 7, SOLUTION 1.

CREATE AN "INCIDENCE MATRIX" WITH 9 ROWS (CORRESPONDING TO PROBLEMS) AND 17 COLUMNS (CORRESPONDING TO CONTESTANTS).

$$
\begin{array}{c|ccccc|}
 & c_1 & c_2 & c_3 & & c_{17} \\
\hline
p_1 & 1 & 1 & 0 & & 1 \\
p_2 & 0 & 1 & 0 & & 0 \\
\vdots & \vdots & & & \cdots & \\
p_9 & 0 & 0 & 1 & & 1 \\
\hline
\end{array}
$$

PLACE A "1" IN THE MATRIX IF THAT CONTESTANT SOLVED THAT PROBLEM, AND PLACE A "0" OTHERWISE.

SUPPOSE THAT THE RESULT TO BE PROVED IS FALSE.
THEN EVERY PAIR OF COLUMNS (c_i, c_j) HAS AT LEAST ONE OCCURRENCE OF $[0\ 0]$.

LET'S COUNT ALL "ROW ZERO PAIRS" IN TWO WAYS.

FIRST, THERE ARE 6 ZEROS IN EACH ROW

$\Rightarrow \binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ ZERO PAIRS PER ROW

$\Rightarrow$ TOTAL $= 9 \cdot 15 = \underline{135}$ ROW ZERO PAIRS.

IF THE RESULT TO BE PROVED IS FALSE, THE NUMBER OF ROW ZERO PAIRS IS AT LEAST

$$\binom{17}{2} = \frac{17 \cdot 16}{2} = 17 \cdot 8 = \underline{136}$$

CONTRADICTION!

THUS THE RESULT IS PROVED. $\underline{\text{QED}}$.

EXAMPLE 7, SOLUTION 2.

AGAIN WE CREATE AN INCIDENCE MATRIX, BUT THIS TIME

ROWS = PROBLEMS

COLUMNS = PAIRS OF CONTESTANTS.

$\Rightarrow$ THERE ARE 9 ROWS AND $\binom{17}{2} = 136$ COLUMNS.

DENOTE THE CONTESTANT PAIRS BY $g_1, g_2, \dots, g_{136}$.

$$
\begin{array}{c c}
 & \begin{array}{cccccc} g_1 & g_2 & g_3 & & & g_{136} \end{array} \\
\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_9 \end{array} &
\left[\begin{array}{c|c|c|ccc|c}
1 & 1 & 0 & & & & 1 \\ \hline
0 & 1 & 1 & & & & 0 \\ \hline
 & & & & \cdots & & \\ \hline
0 & 1 & 0 & & & & 0
\end{array}\right]
\end{array}
$$

FOR EACH ENTRY, PLACE A "1" IF AT LEAST ONE OF THE CONTESTANTS FROM g_j SOLVED PROBLEM p_i [(i,j) ENTRY].

FOR EACH PROBLEM p_i, THE NUMBER OF CONTESTANT PAIRS WHERE BOTH FAILED TO SOLVE p_i IS $\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$.

$\Rightarrow$ THE NUMBER OF 1'S PER ROW IS

$$136 - 15 = 121.$$

$\Rightarrow$ TOTAL NO. OF 1'S IN THE MATRIX IS

$$121 \cdot 9 = \underline{1089}.$$

NOW, COUNT THE 1'S BY COLUMNS: IF THE RESULT TO BE PROVED IS FALSE, THERE ARE AT MOST 8 1S IN ANY COLUMN

$\Rightarrow$ TOTAL NO. OF 1'S IN MATRIX $\leq 8 \cdot 136 = \underline{1088}$

CONTRADICTION!

QED.

EXAMPLE 2.

CREATE AN INCIDENCE MATRIX

ROWS = CONTESTANTS

COLUMNS = PAIRS OF JUDGES

NOTE THERE ARE α ROWS AND $n = \binom{6}{2}$ COLUMNS.

DENOTE THE PAIRS OF JUDGES BY

$$g_1, g_2, \ldots, g_n.$$

$$\begin{array}{c} \\ c_1 \\ c_2 \\ \vdots \\ c_\alpha \end{array} \begin{array}{c} \begin{array}{cccc} g_1 & g_2 & & g_n \end{array} \\ \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ \hline 0 & 1 & & 1 \\ \hline & & \cdots & \\ \hline 1 & 1 & & 1 \end{array}\right] \end{array}$$

PLACE A "1" IN THE (i,j)-ENTRY OF THE MATRIX IF THE PAIR OF JUDGES g_j AGREE ON THE RATING FOR CONTESTANT c_i, AND PLACE A "0" OTHERWISE.

THEN, k IS THE MAXIMUM NO. OF 1'S IN ANY COLUMN OF THE MATRIX.

$\Rightarrow$ NO. OF 1'S IN THE MATRIX IS AT MOST

$k\binom{b}{2}$. $\circledast$

NOW, LET'S COUNT THE NO. OF 1'S IN THE MATRIX ROW BY ROW.

CONSIDER ANY ROW (CONTESTANT).

WITH b RATINGS, OF WHICH r ARE "PASS" AND $(b-r)$ ARE "FAIL", WE GET

$$\binom{r}{2} + \binom{b-r}{2} \quad \text{COINCIDENCES}$$

THIS IS THE NO. OF 1'S IN THAT ROW OF THE MATRIX.

HOW SMALL CAN THIS NUMBER BE?

<u>USEFUL RESULT</u>:

IF $a \leq b$, THEN FOR ANY $x \geq 0$,

[HERE $a, b, x \in \mathbb{Z}$]

$$\binom{a-x}{2} + \binom{b+x}{2} \geq \binom{a}{2} + \binom{b}{2}$$

WITH EQUALITY IF AND ONLY IF $x = 0$.

PROOF:

$$\binom{a-x}{2} + \binom{b+x}{2} \geqslant \binom{a}{2} + \binom{b}{2}$$

$$\Leftrightarrow (a-x)(a-x-1) + (b+x)(b+x-1) \geqslant a(a-1) + b(b-1)$$

$$\Leftrightarrow -ax - x(a-x-1) + bx + x(b+x-1) \geqslant 0$$

$$\Leftrightarrow -2ax + x^2 + x + 2bx + x^2 - x \geqslant 0$$

$$\Leftrightarrow 2(b-a)x + 2x^2 \geqslant 0$$

THIS IS CLEARLY TRUE SINCE $a \leqslant b$ & $x \geqslant 0$.

ALSO, FOR EQUALITY WE NEED $x = 0$. QED.

CONCLUSION: IF a AND b ARE AT LEAST 2 APART, THEN MOVING THEM CLOSER TOGETHER STRICTLY DECREASES $\binom{a}{2} + \binom{b}{2}$.

THEREFORE, SINCE b IS ODD, THE MINIMUM NO. OF COINCIDENCES AMONG THE b JUDGES IS

$$\binom{(b-1)/2}{2} + \binom{(b+1)/2}{2}$$

$$= \frac{1}{2}\left(\frac{b-1}{2}\right)\left(\frac{b-1}{2}-1\right) + \frac{1}{2}\left(\frac{b+1}{2}\right)\left(\frac{b+1}{2}-1\right)$$

$$= \frac{1}{8}\left[(b-1)(b-3) + (b+1)(b-1)\right]$$

$$= \frac{1}{8}(b-1)(2b-2)$$

$$= \frac{1}{4}(b-1)^2 .$$

THIS IS FOR A SINGLE CONTESTANT (MIN NO. OF 1'S PER ROW OF THE MATRIX).

$\Rightarrow$ TOTAL NO. OF 1'S IN THE MATRIX IS AT LEAST

$$\frac{a}{4}(b-1)^2 \qquad (**)$$

THUS (COMBINING (*) AND (**)) WE NEED

$$k\binom{b}{2} \geq \frac{a}{4}(b-1)^2$$

i.e.,

$$k\,\frac{b(b-1)}{2} \geq \frac{a}{4}(b-1)^2$$

SIMPLIFYING, WE GET

$$\underline{\frac{k}{a} \geq \frac{b-1}{2b}}\,.$$

QED.

ROUGH WORK

NO. OF WAYS OF CHOOSING 2 OBJECTS OUT OF n OBJECTS IS

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

e.g. $\binom{4}{2} = \frac{4(3)}{2} = 6$

ABCD

AB
AC
AD
BC
BD
CD $\Rightarrow$ 6 WAYS

e.g. 6 = 7 judges

r SAY "PASS"

7 − r SAY "FAIL"

$r=0 \quad \binom{0}{2} + \binom{7}{2} = 21$

$r=1 \quad \binom{1}{2} + \binom{6}{2} = 0 + 15 = 15$

$r=2 \quad \binom{2}{2} + \binom{5}{2} = 1 + 10 = 11$ PPFFFFF

$r=3 \quad \binom{3}{2} + \binom{4}{2} = 3 + 6 = 9$ PPPFFFF

$r=4 \quad \binom{4}{2} + \binom{3}{2} = 6 + 3 = 9$

$r=5 \quad \binom{5}{2} + \binom{2}{2} = 10 + 1 = 11$

$r=6 \quad \binom{6}{2} + \binom{1}{2} = 15 + 0 = 15$

$r=7 \quad \binom{7}{2} + \binom{0}{2} = 21$