MATHEMATICAL ENRICHMENT / OLYMPIAD TRAINING

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INEQUALITIES

BASIC INEQUALITY : $x^2 \geq 0$ x is a real number

Let's apply this to $(x-y)^2 \geq 0$

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy.$$

Example Show that

$$x^2 + y^2 + z^2 \geq xy + yz + xz$$

where x, y, z are real numbers.

Solution:

$$x^2 + y^2 \geq 2xy$$

$$y^2 + z^2 \geq 2yz$$

$$x^2 + z^2 \geq 2xz$$

Add $2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2xz$

divide by 2.

Consider $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$ $x \geq 0$, $y \geq 0$

$$x - 2\sqrt{x}\sqrt{y} + y \geq 0$$

$$x + y \geq 2\sqrt{xy}$$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Arithmetic mean of x, y

Geometric mean of x, y.

AM-GM inequality

Example If $x > 0$ show $x + \frac{1}{x} \geq 2$

Solution: Apply AM-GM inequality to x and $\frac{1}{x}$

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

Multiply by 2 : $x + \frac{1}{x} \geq 2$.

Example If $x, y > 0$ show $\frac{x}{y} + \frac{y}{x} \geq 2$

Apply AM-GM to $\frac{x}{y}$ and $\frac{y}{x}$

Example Show $(a+b)(b+c)(a+c) \geq 8abc$ $\quad a>0, \; b>0, \; c>0$

Solution:

$a+b \geq 2\sqrt{ab}$ by AM-GM
$b+c \geq 2\sqrt{bc}$ " "
$a+c \geq 2\sqrt{ac}$ " "

Multiply

$$(a+b)(b+c)(a+c) \geq 8\sqrt{ab}\sqrt{bc}\sqrt{ac} = 8\sqrt{a^2b^2c^2} = 8abc.$$

Let $a_1, a_2, \ldots, a_n$ be real numbers ≥ 0

Then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

AM-GM inequality

<u>Example</u> Show $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3$

$x>0$
$y>0$
$z>0$

Apply AM-GM for 3 numbers to $\frac{x}{y}$, $\frac{y}{z}$, $\frac{z}{x}$

get $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq 3\sqrt[3]{\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}} = 3$

<u>Example</u> Show $x^6+y^6+4 \geq 6xy$

$x>0$
$y>0$

Apply AM-GM for 6 numbers to $x^6, y^6, 1, 1, 1, 1$

$$\Rightarrow \quad x^6+y^6+1+1+1+1 \geq 6\sqrt[6]{x^6\cdot y^6\cdot 1\cdot 1\cdot 1\cdot 1}$$
$$= 6xy.$$

<u>Example</u> Show $x^2+\frac{4}{x} \geq 3\sqrt[3]{4}$

Apply AM-GM for 3 numbers to
$x^2, \frac{2}{x}, \frac{2}{x}$

$$x^2+\frac{2}{x}+\frac{2}{x} \geq 3\cdot\sqrt[3]{x^2\cdot\frac{2}{x}\cdot\frac{2}{x}}$$
$$= 3\sqrt[3]{4}$$

Apply AM-GM to $\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}$ $a_i > 0$

get
$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}}$$
$$= \frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}}$$

So
$$\underbrace{\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}}_{\text{Harmonic Mean of } a_1, a_2, \dots, a_n} \leq \sqrt[n]{a_1 a_2 \cdots a_n}$$

So $H.M. \leq G.M. \leq A.M.$

Example Show $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ when $a+b+c=1$, $a>0, b>0, c>0$

Apply $H.M. \leq A.M.$ for $n=3$

$$\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \leq \frac{a+b+c}{3} = \frac{1}{3}$$

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \geq 3$$

Multiply by 3

H.W. $$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$