THE PRINCIPLE OF INDUCTION

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The Principle of Induction: Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The principle of induction is a way of proving that $P(n)$ is true for all integers $n \geq a$. It works in two steps:

(a) [**Base case:**] Prove that $P(a)$ is true.
(b) [**Inductive step:**] Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

This principle is very useful in problem solving, especially when we observe a *pattern* and want to prove it.

The trick to using the Principle of Induction properly is to spot *how to use* $P(k)$ to prove $P(k+1)$. Sometimes this must be done rather ingeniously!

Problem 1. Prove that for any integer $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \, .$$

Solution. Let $P(n)$ denote the proposition to be proved. First let's examine $P(1)$: this states that

$$1 = \frac{1(2)}{2} = 1$$

which is correct.

Next, we assume that $P(k)$ is true for some positive integer k, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \, .$$

and we want to use this to prove $P(k+1)$, i.e.

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} \, .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Problem 2. Find a formula for the sum of the first n odd numbers.

Solution. Note that this time we are not told the formula that we have to prove; we have to find it ourselves! Let's try some small numbers and see if a pattern emerges:

$$1 = 1; \quad 1 + 3 = 4; \quad 1 + 3 + 5 = 9;$$
$$1 + 3 + 5 + 7 = 16; \quad 1 + 3 + 5 + 7 + 9 = 25;$$

We conjecture (guess) that the sum of the first n odd numbers is equal to n^2. Now let's prove this proposition using the principle of induction; call it $P(n)$.
Our statement $P(n)$ is that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 .$$

First we prove the base case $P(1)$, i.e.

$$1 = 1^2$$

This is certainly true. Now we assume that $P(k)$ is true, i.e.

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2 .$$

and consider $P(k + 1)$:

$$1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2 .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= (1+3+5+\cdots+(2k-1))+(2k+1) \\ &= k^2+(2k+1) \\ &= (k+1)^2 . \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Exercise 1. Show that for all $n \geq 1$,

$$1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(4n^2-1)}{3} .$$

Exercise 2. Show that for all $n \geq 1$, we have $f(n) = g(n)$, where

$$f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

and

$$g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} .$$

Problem 3. Show that 6 divides $8^n - 2^n$ for every positive integer n.

Solution. We will use induction. First we prove the base case $n = 1$, i.e. that 6 divides $8^1 - 2^1 = 6$; this is certainly true.

Next assume that proposition holds for some positive integer k, i.e. 6 divides $8^k - 2^k$. Let's examine $8^{k+1} - 2^{k+1}$:

$$\begin{aligned} 8^{k+1} - 2^{k+1} &= 8 \cdot 8^k - 2 \cdot 2^k \\ &= 6 \cdot 8^k + 2 \cdot 8^k - 2 \cdot 2^k \\ &= 6 \cdot 8^k + 2 \cdot \left(8^k - 2^k\right) \ . \end{aligned}$$

Now since 6 divides $8^k - 2^k$ (by assumption), and 6 certainly divides $6 \cdot 8^k$, it follows that 6 divides $8^{k+1} - 2^{k+1}$. Therefore by the principle of induction, 6 divides $8^n - 2^n$ for every positive integer n.

Exercise 3. For every $n \geq 1$, define

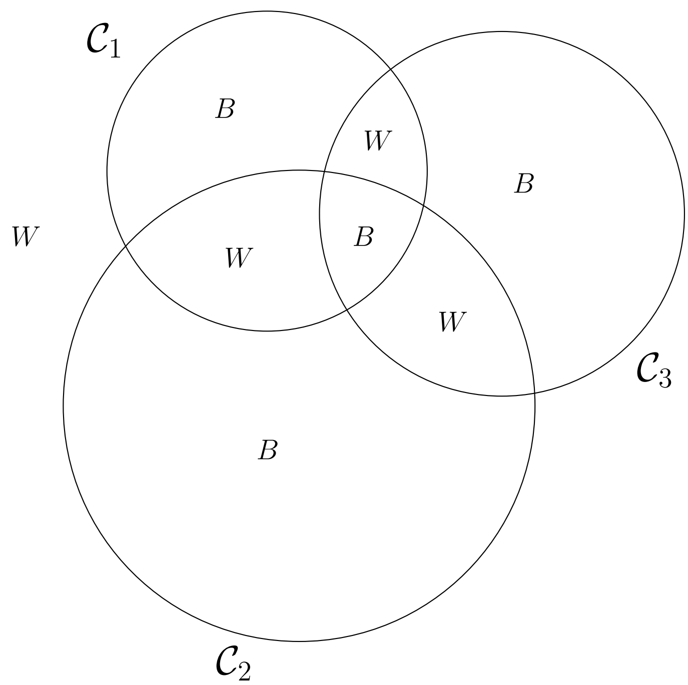
$$S(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \ .$$

Show that $S(n)$ is an integer for every $n \geq 1$.

Problem 4. $n \geq 1$ circles are given in the plane. They divide the plane into *regions*. Show that it is possible to colour the plane using two colours, so that no two regions with a common boundary line are assigned the same colour.

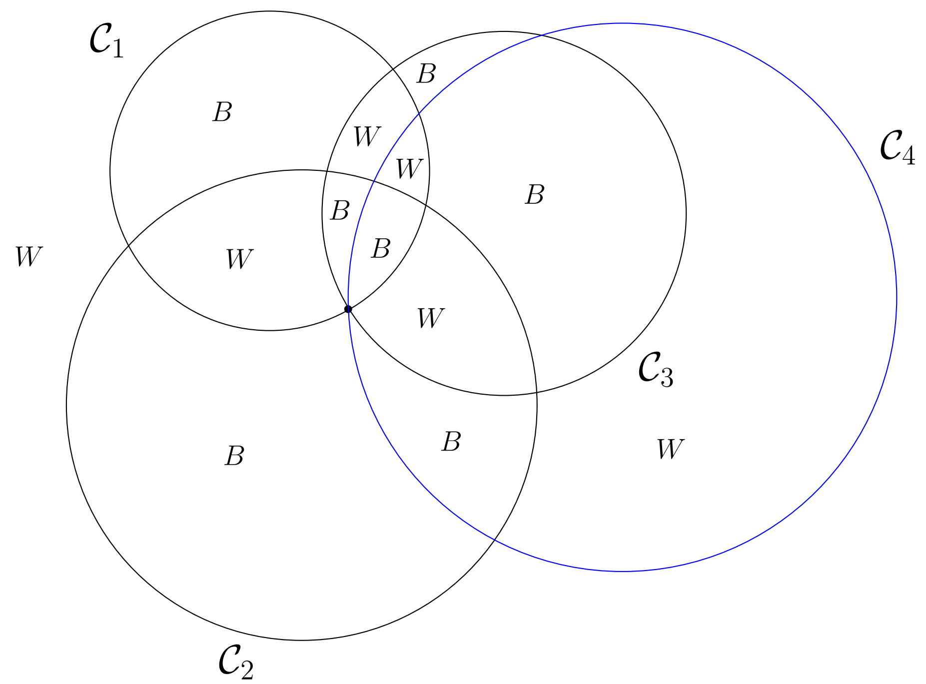
Solution. Call the proposition $P(n)$. Let the two colours be B and W. For $n = 1$, the result $P(1)$ is clear; if there is only one circle, we may colour the inside B and the outside W, and this colouring satisfies the conditions of the problem.

Assume the result $P(n)$ holds for $n = k$ circles; so we know that for any k circles there is a colouring which satisfies the conditions of the problem. An example for 3 circles is shown below.



Next consider $k+1$ circles $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{k+1}\}$. Ignoring the circle $\mathcal{C}_{k+1}$, we now have k circles $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k\}$. By the $P(k)$ assumption, there is a colouring of the plane which satisfies the conditions of the problem; we colour the plane according to this colouring.

Now we add the circle $\mathcal{C}_{k+1}$ back into the picture, as shown below for the example at hand:

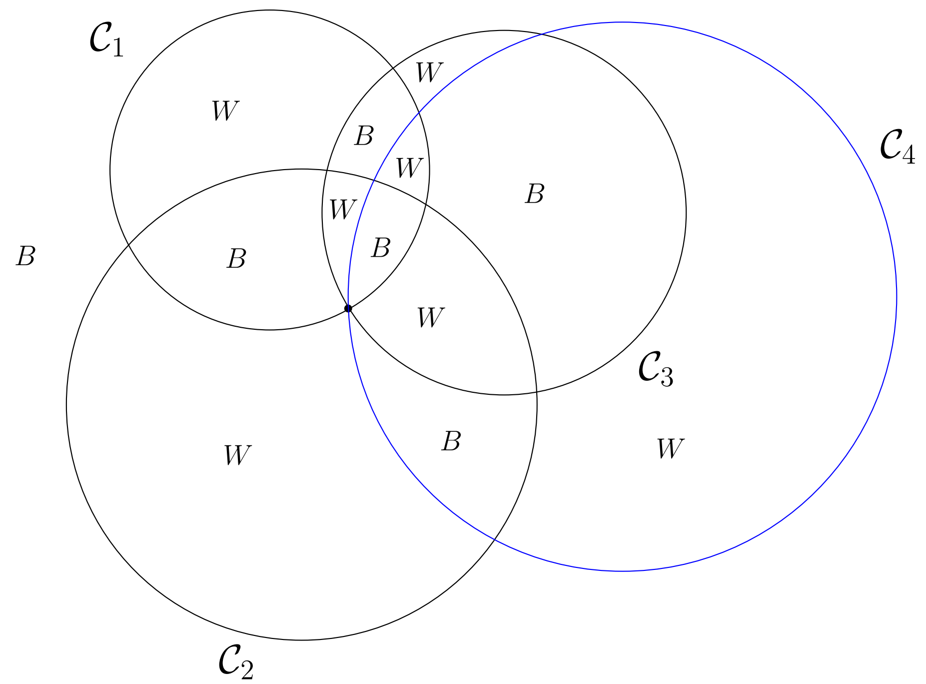


To obtain a new colouring, we do the following:

(a) for any region which lies *inside* $\mathcal{C}_{k+1}$, *do not change* its colour.

(b) for any region which lies *outside* $\mathcal{C}_{k+1}$, recolour it into the *opposite* colour.

The result of this colouring is shown below for the example at hand:



Now we may check that the new colouring works:

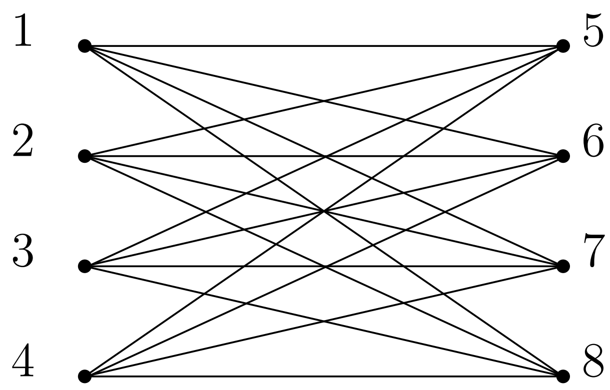
(i) two neighbouring regions whose boundary lies *inside* $\mathcal{C}_{k+1}$ have different colours (by $P(k)$ assumption);
(ii) two neighbouring regions whose boundary lies *outside* $\mathcal{C}_{k+1}$ have different colours (by $P(k)$ assumption, and the fact that we re-coloured *both* colours on either side of the boundary);
(iii) two neighbouring regions whose boundary lies *on* $\mathcal{C}_{k+1}$ have different colours (due to the fact that these colours were the same initially, and *one* of them was then recoloured).

This shows that $P(k+1)$ is true, and so by the principle of induction, the proof follows.

Exercise 4.

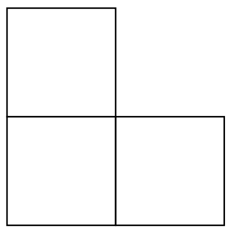
$2n$ points are given in space, where $n \geq 2$. Altogether $n^2 + 1$ line segments ('edges') are drawn between these points. Show that there is at least one set of three points which are joined pairwise by line segments (i.e. show that there exists a *triangle*).

Note. If we have $2n$ points and *exactly* n^2 edges, it is possible to *avoid* making a triangle. This is done by breaking the set of points into two subsets $\mathcal{X}$ and $\mathcal{Y}$ which contain n points each, then connecting every point in $\mathcal{X}$ to every point in $\mathcal{Y}$. This is illustrated below for the case $n = 4$.



Exercise 5.

Let n be a positive integer. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, the remaining board can be tiled with 3-square tiles of the following shape:



Exercise 6.

Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet at a single point (e.g. $f(1) = 2$; $f(2) = 4$; etc.). Find a formula for $f(n)$.

Exercise 7. Every road in Uniland is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly of via at most one other city.

Pólya's Paradox:

A common way (in 1950, at least!) of expressing that something is out of the ordinary is "*That's* a horse of a different color!" The famous mathematician George Pólya gave the following proof that "all horses are the same color", which works by the principle of induction:

Proposition $P(n)$: Suppose we have n horses. Then all n horses are the same colour.

Base case: $n = 1$; if there is only one horse, there is only one colour.

Inductive step: Assume that $P(k)$ is true, i.e. that for any set of k horses, there is only one color. Now look at any set of $k + 1$ horses; call this $\{H_1, H_2, H_3, \cdots, H_k, H_{k+1}\}$. Consider the sets $\{H_1, H_2, H_3, \cdots, H_k\}$ and $\{H_2, H_3, H_4, ..., H_{k+1}\}$. Each is a set of only k horses, therefore within each there is only one colour. But the two sets overlap, so there must be only one colour among all $k + 1$ horses.

The flaw is that when $k = 2$ the inductive step doesn't work, because the statement that "the two sets overlap" is false.