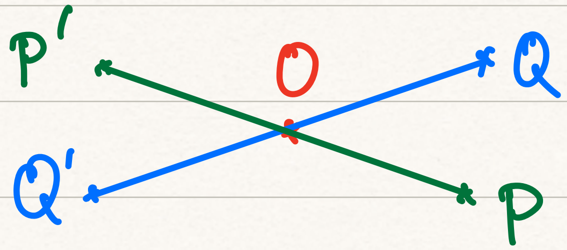
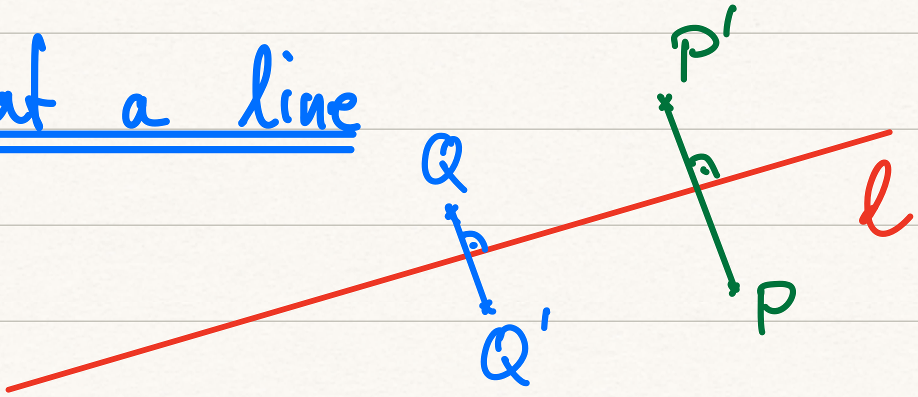
Circle Inversion

We know how to

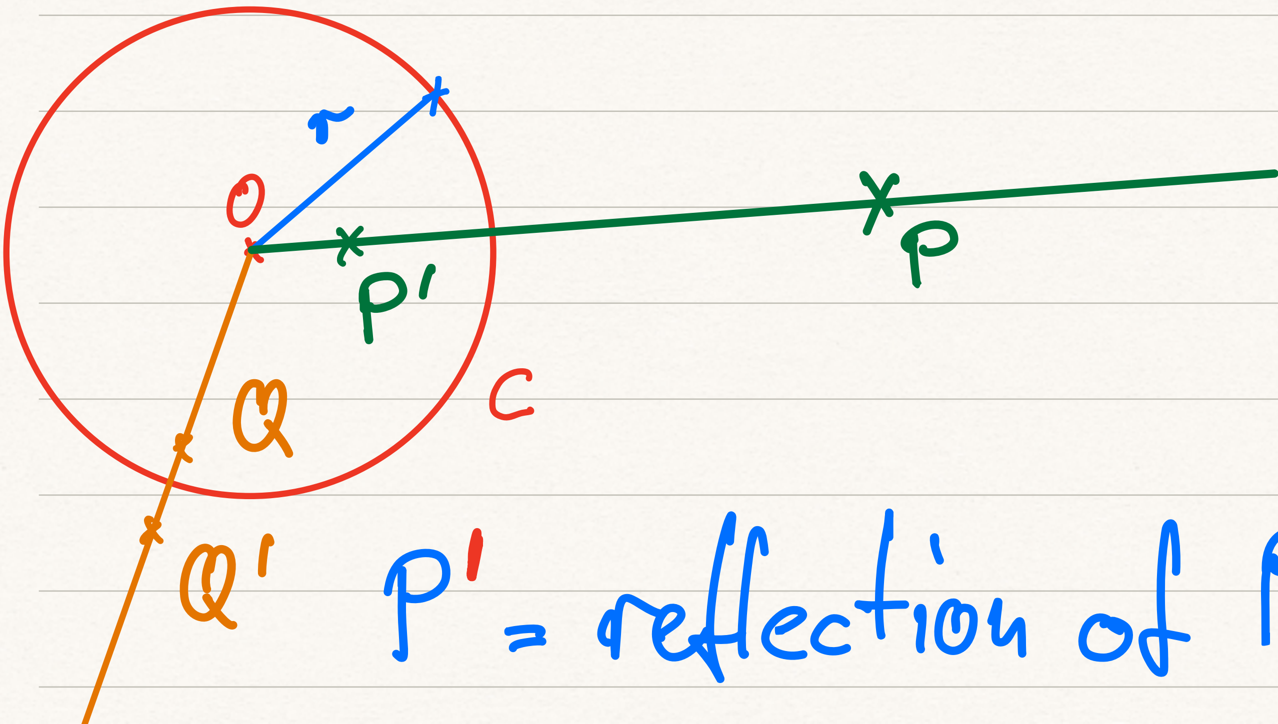
(i) reflect at a point



(ii) reflect at a line



Why not reflect in a circle?



P' = reflection of P in C

How should $P \mapsto P'$ look like?

(i) points on circle are invariant

(ii) O, P, P' are collinear.

(iii) If $P \to$ circle,
then P and P' are close.

(iv) If $P \to O$, then $P' \to \infty$
If $P \to \infty$, then $P' \to O$

(v) reflection should be
self-inverse: $(P')' = P$

(vi) O is not defined,
or set $O' = $ "∞".

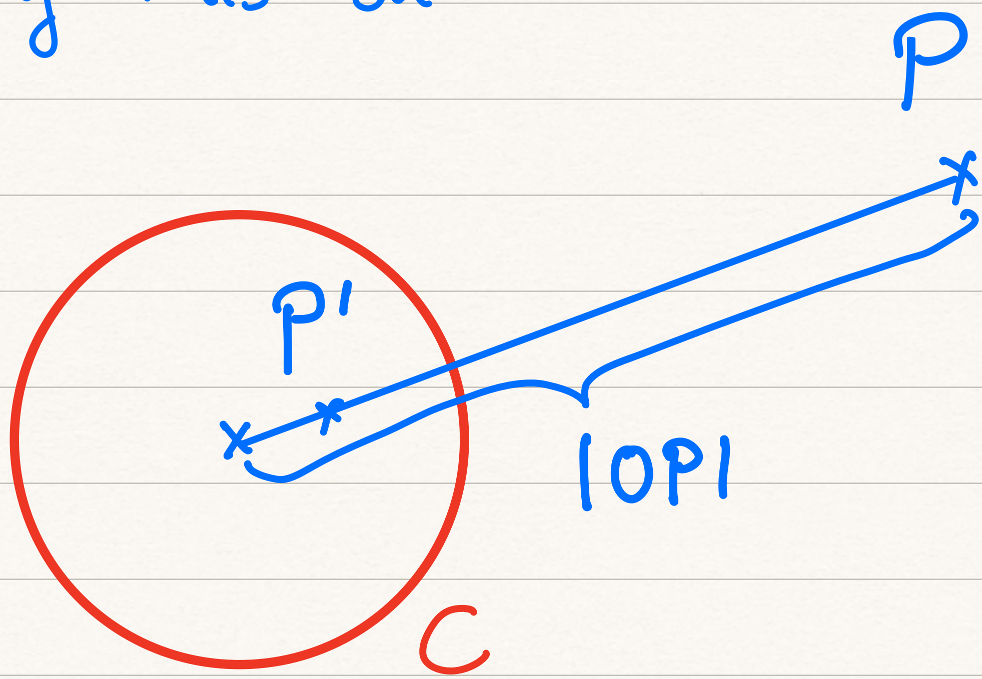
This is all satisfied by

$P \mapsto P'$ = unique point on the ray $\overrightarrow{OP}$ s.t.

$$|OP| \cdot |OP'| = r^2.$$

Let us try this out!

$$|OP| \cdot |OP'| = r^2$$



1 Basic properties

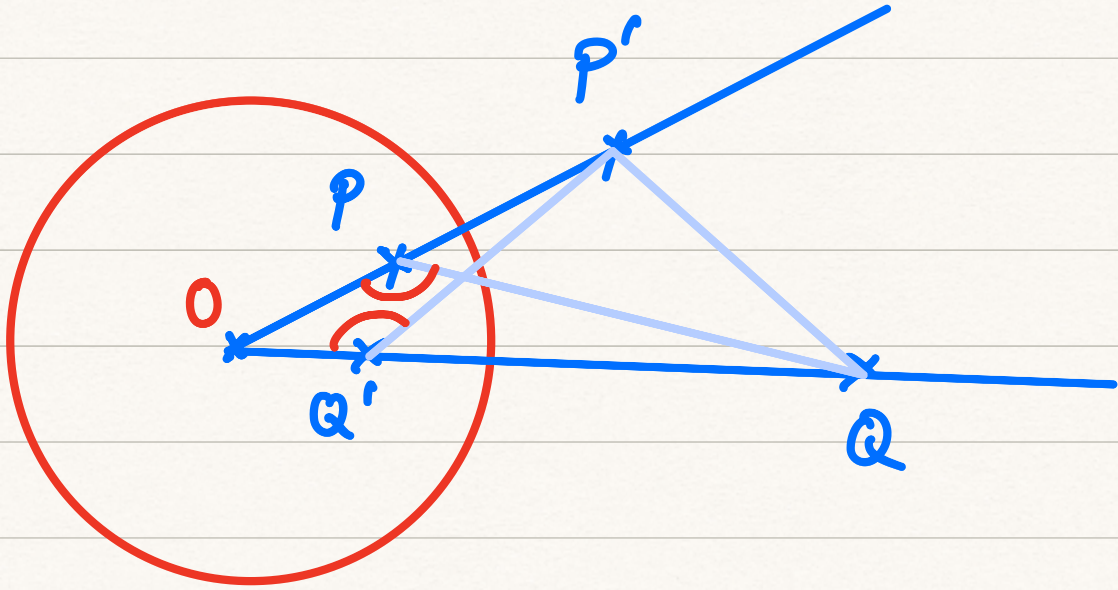
(0) distances are <u>not</u> preserved

(1) $(P')' = P$

(2) P inside $C \iff P'$ outside C

Points on C are invariant.

(3) Some angles are "preserved":



$$\sphericalangle OPQ = \sphericalangle P'Q'O$$

below: (anti-) conformality.

<u>Proof</u>: The triangles

$\triangle POQ$ and $\triangle Q'OP'$

are <u>similar</u> as

(i) $\measuredangle QOP = \measuredangle Q'OP'$

and

(ii) $|OP| \cdot |OP'| = r^2 = |OQ| \cdot |OQ'|$

$$\Rightarrow \frac{|OP|}{|OQ|} = \frac{|OQ'|}{|OP'|}.$$

Hence, $\measuredangle OPQ = \measuredangle P'Q'O$

as corresponding angles. $\square$

(4) If you know about complex number $\mathbb{C}$:

Reflection in unit circle $|z|=1$ corresponds to

$$z \longmapsto \frac{1}{\bar{z}}$$

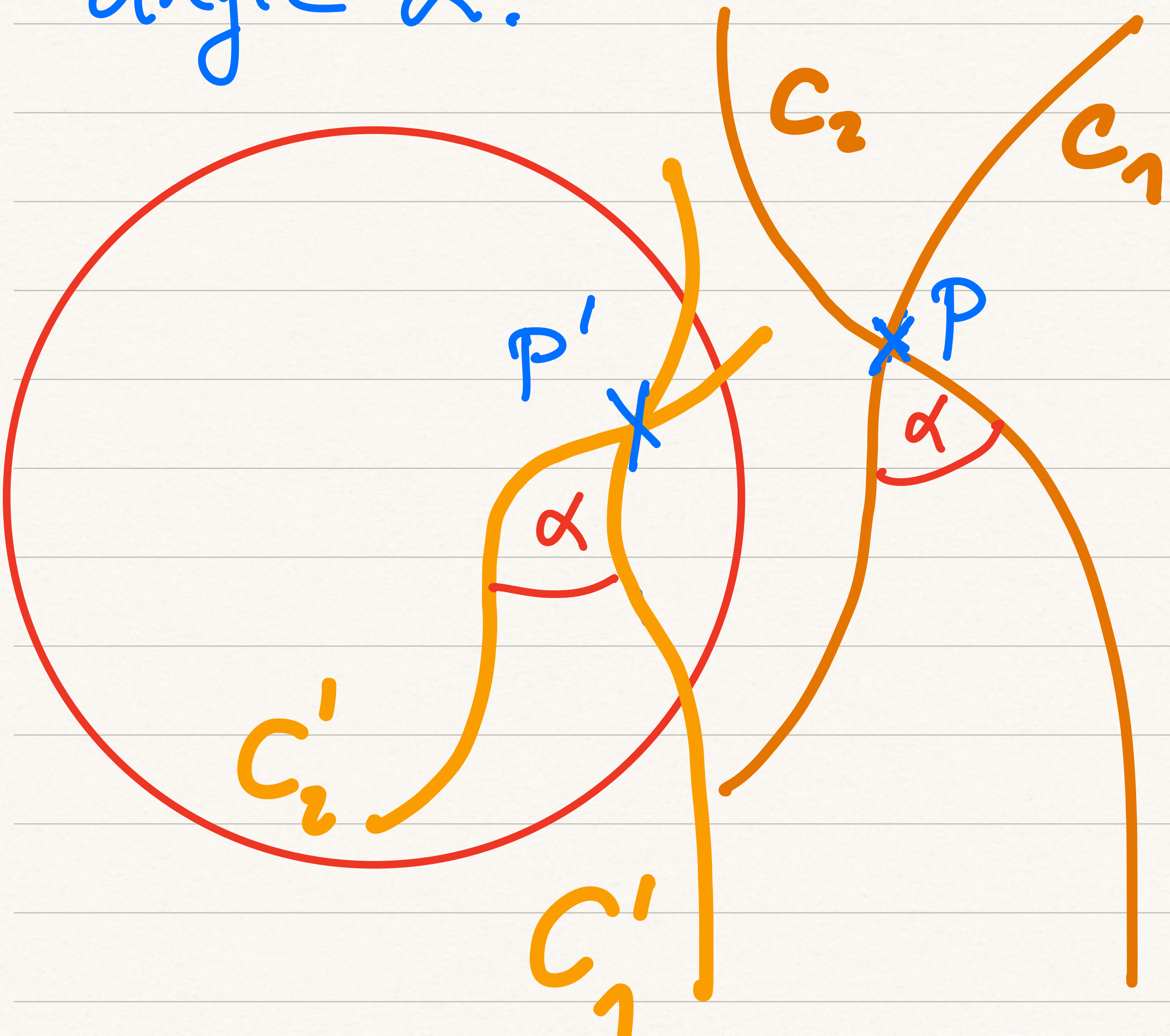
$\bar{z}$ = conjugate of z

② (Anti-) Conformality

In fact, circle inversion preserves all angles between curves (e.g.

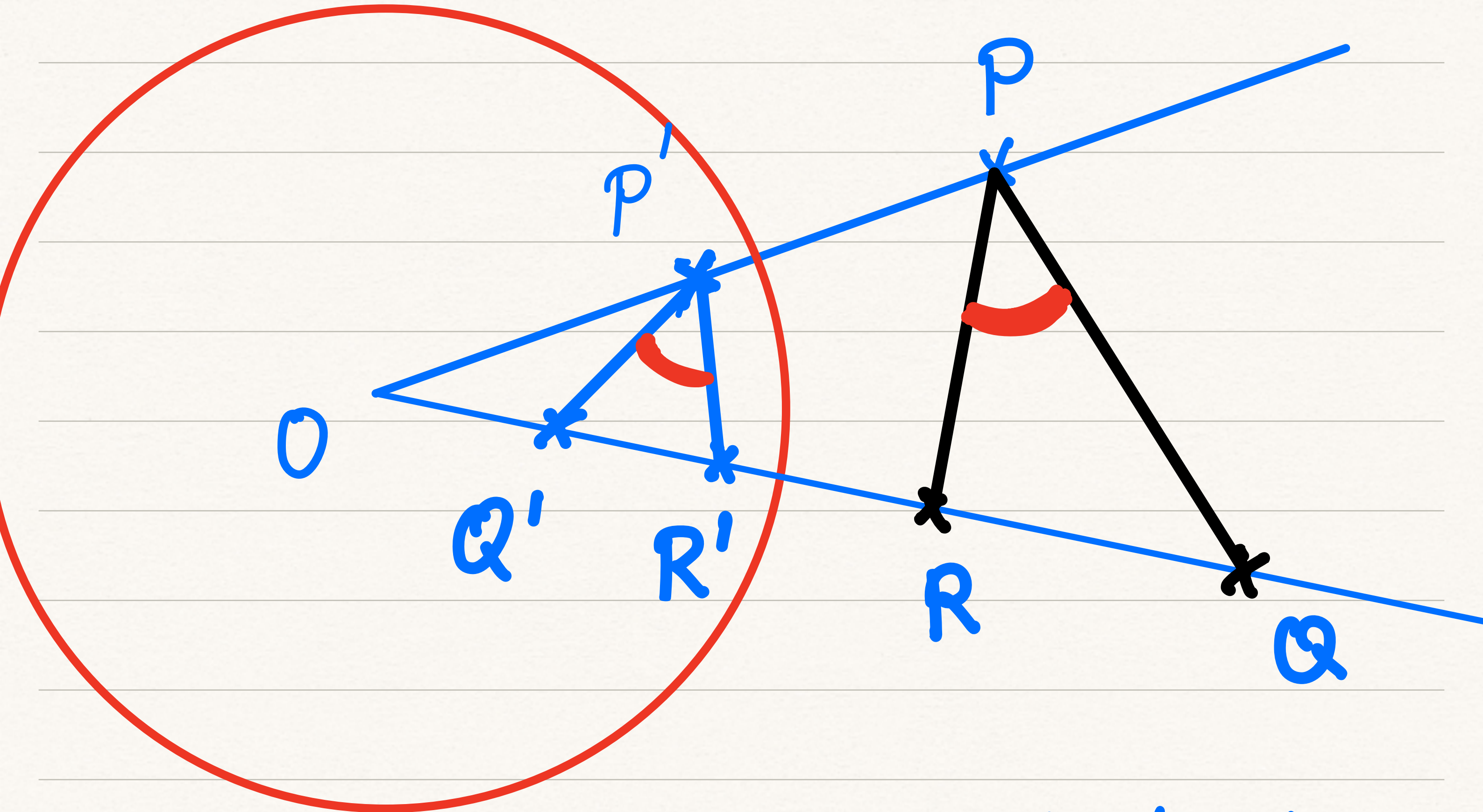
lines, circles, ...)

Proposition: C_1, C_2 curves intersecting in $P \neq O$ with angle α, then C_1', C_2' intersect in P' with angle α.



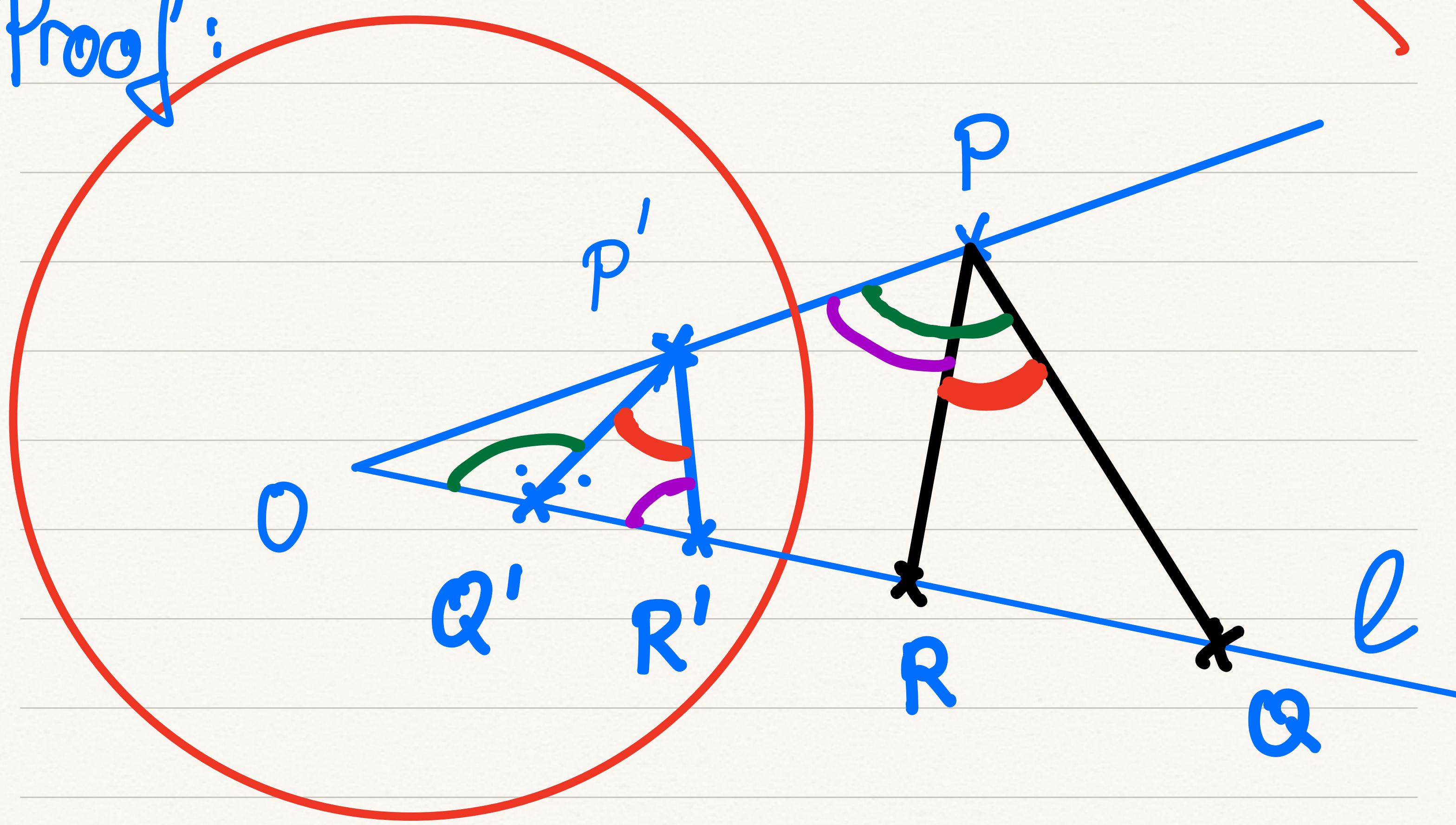
Before its proof, consider the following

Claim (*):



$$\measuredangle RPQ = \measuredangle Q'P'R'$$

Proof:



Proof: $\sphericalangle Q'P'R'$

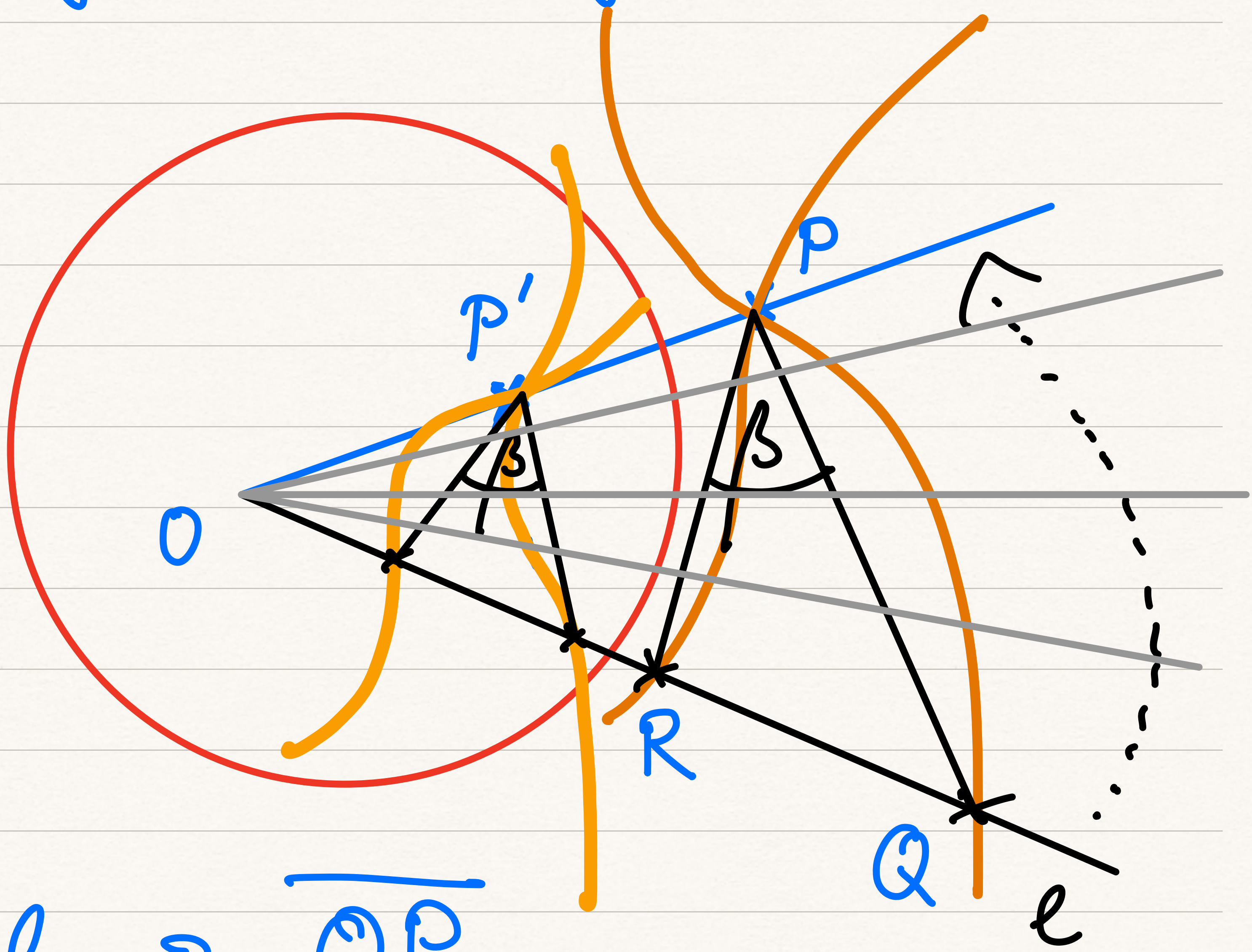
$= 180° - \sphericalangle R'Q'P' - \sphericalangle P'R'Q'$

$= \sphericalangle P'Q'O - \sphericalangle P'R'Q'$

$= \sphericalangle OPQ - \sphericalangle OPR$

$= \sphericalangle RPQ$. $\square$ Claim

The proposition follows by continuity:



$\ell \to \overline{OP}$

Then, $R, Q \to P$

$\beta \to \alpha$.

This shows the proposition. $\square$

③ Clines (= circles + lines) under inversion

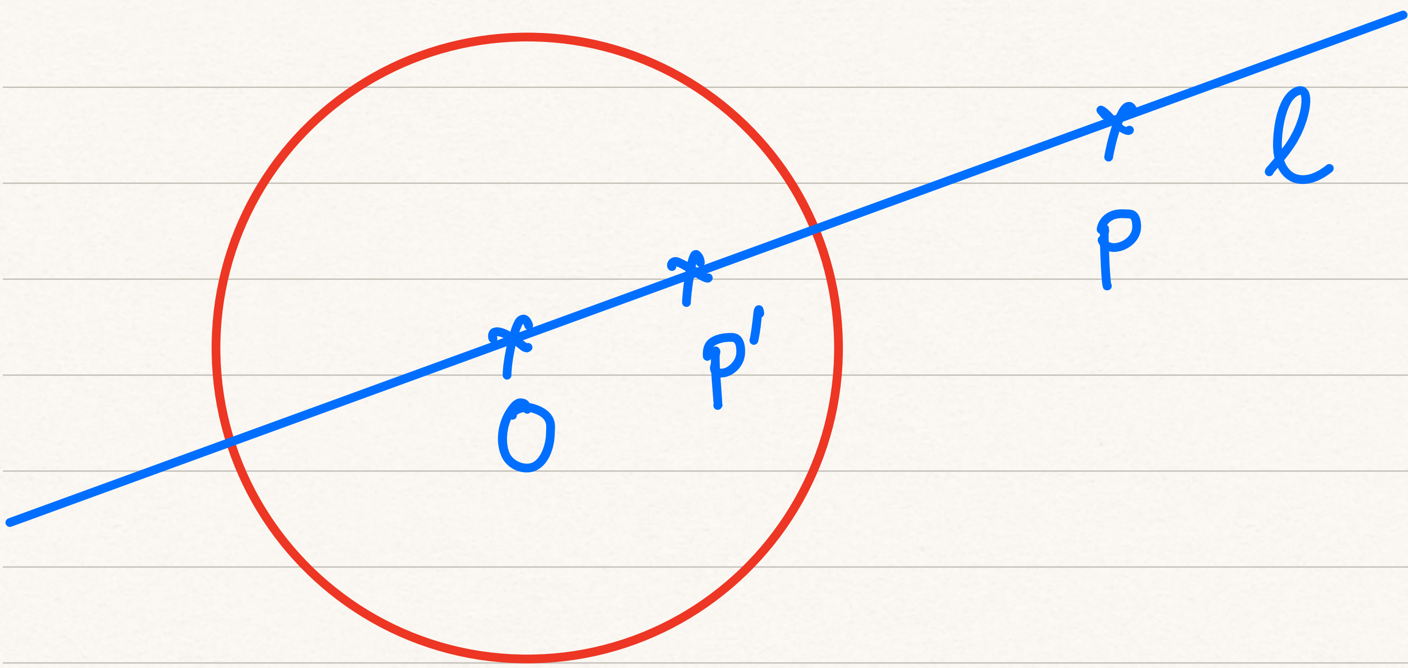
lines $\ni O \longrightarrow$ lines $\ni O$

lines $\not\ni O \longrightarrow$ circles $\ni O$

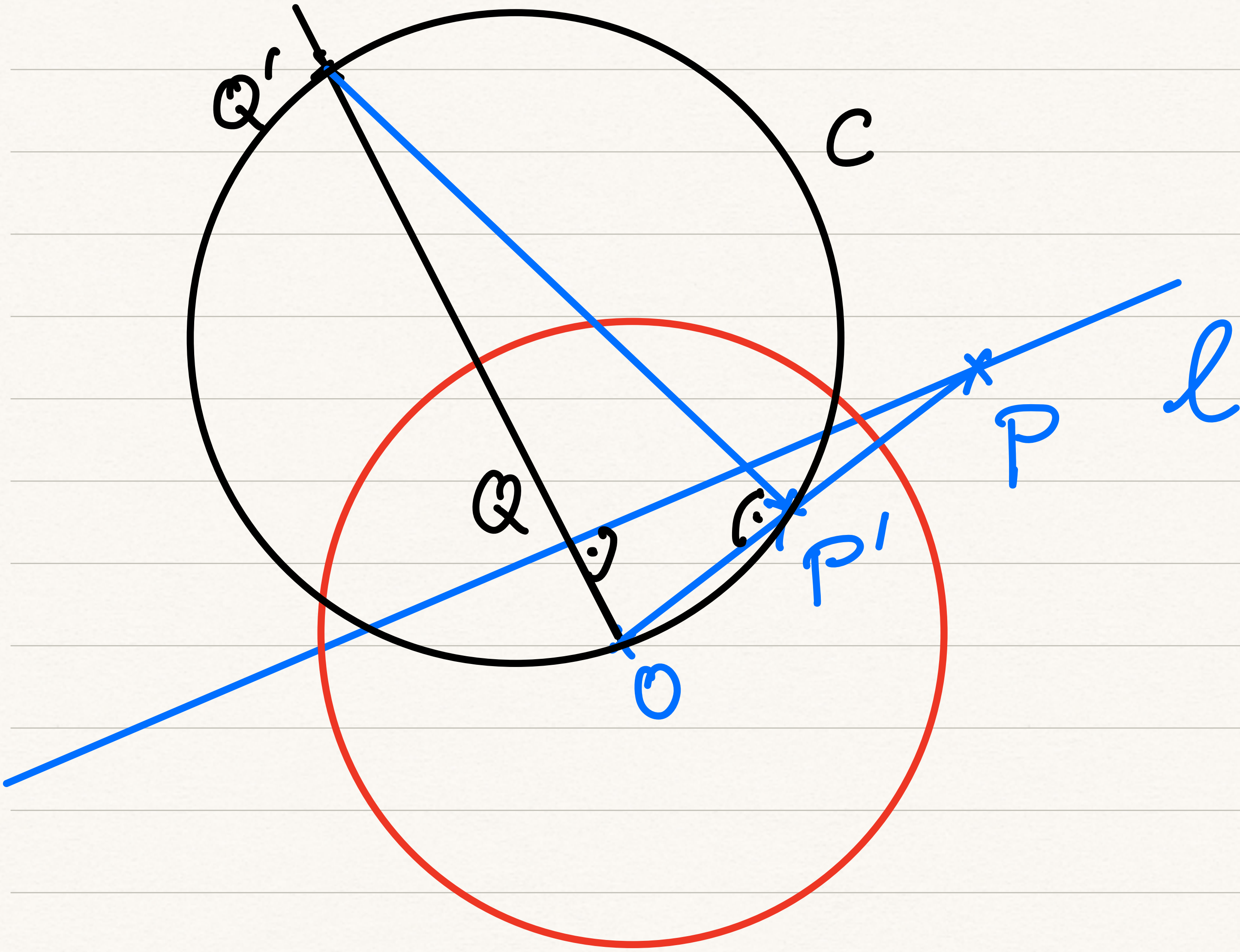
circles $\ni O \longrightarrow$ lines $\not\ni O$

circles $\not\ni O \longrightarrow$ circles $\not\ni O$

<u>(i) line containing O:</u>



(ii) line not containing O:



(1) Q = orthogonal projection of O on ℓ

(2) $\measuredangle Q'P'O = \measuredangle OQP = 90°$

(3) As $\measuredangle Q'PO = 90°$,

P' is on circle C

with diameter $\overline{OQ'}$.

(by Thales's Theorem)

We have proven

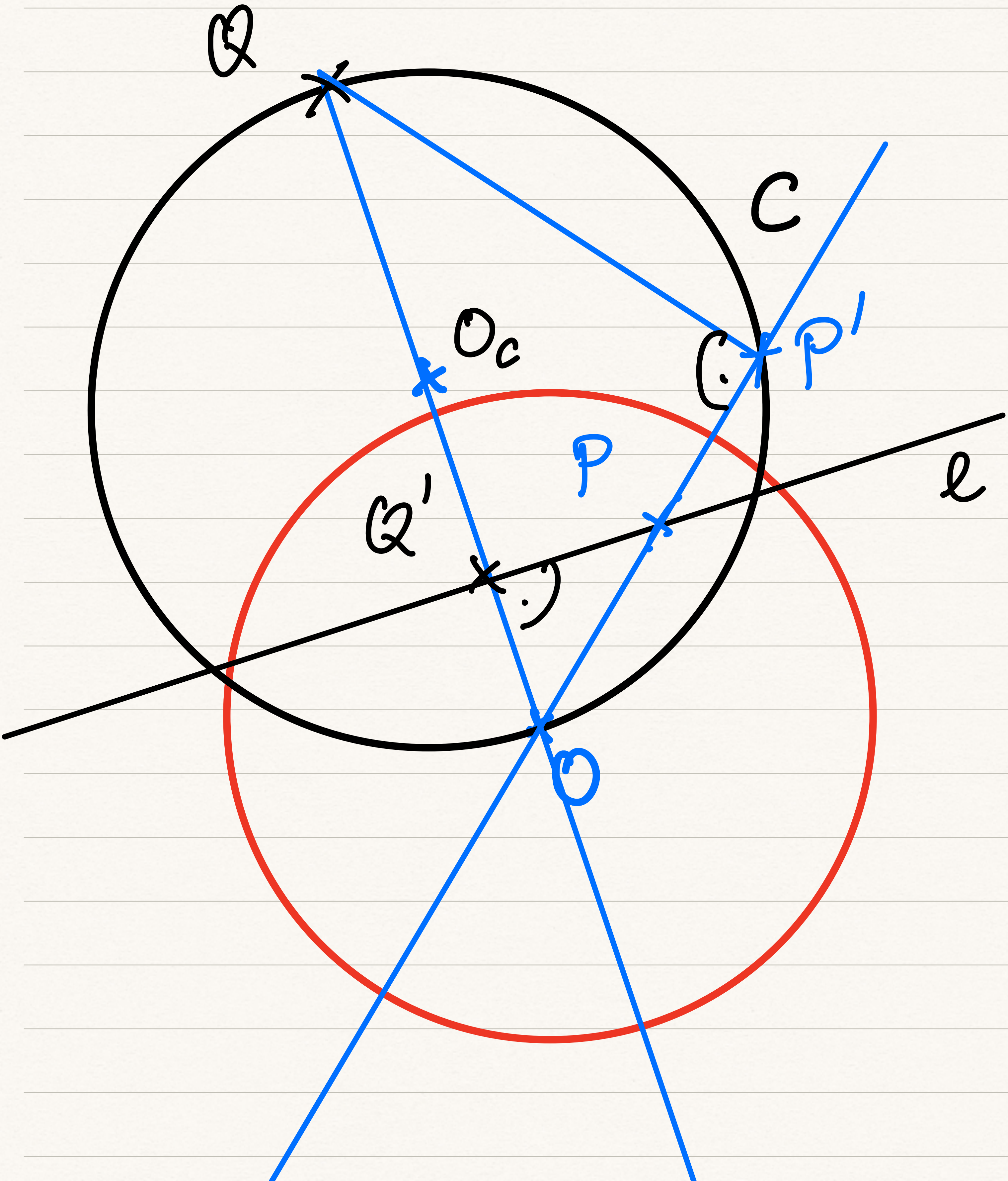
$$P \in \ell \implies P' \in C.$$

(iii) circles through O

This is just converse to (ii).

$$\boxed{\text{Remember: } (P')' = P.}$$

For completeness, here is the argument:



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(1) $Q :=$ second intersection of $\overline{OO_c}$ and C.

(2) $\ell :=$ perpendicular line to $\overline{OO_c}$ at Q'.

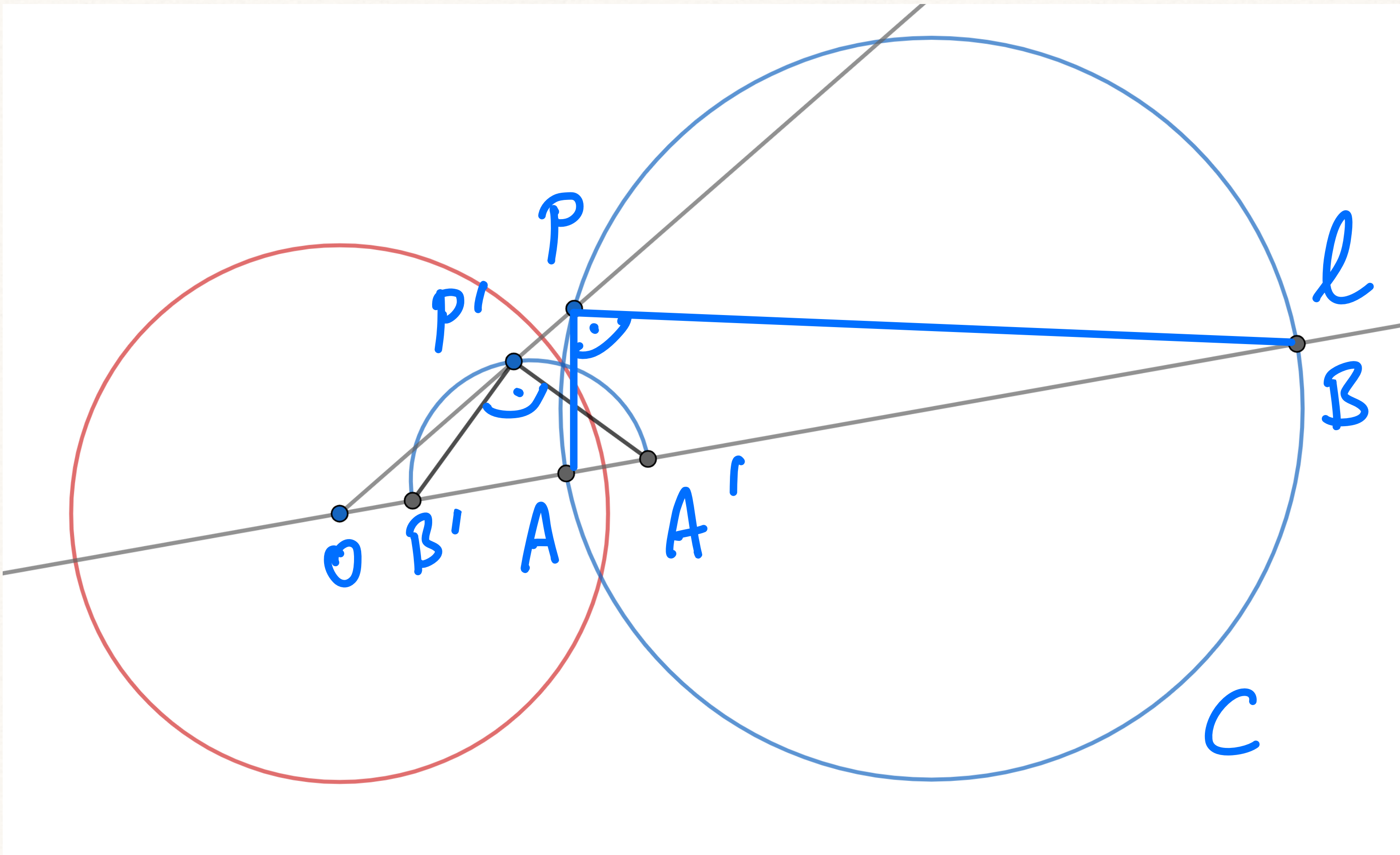
Not surprisingly:

$$\sphericalangle QP'O = \sphericalangle OQ'P = 90°.$$

(3) Thales: $P' \in C$

□.

(iv) circles not through O:



(1) Select ℓ intersecting C in diameter AB

(2) Above Claim (*):

$\sphericalangle B'P'A' = \sphericalangle APB = 90°$.

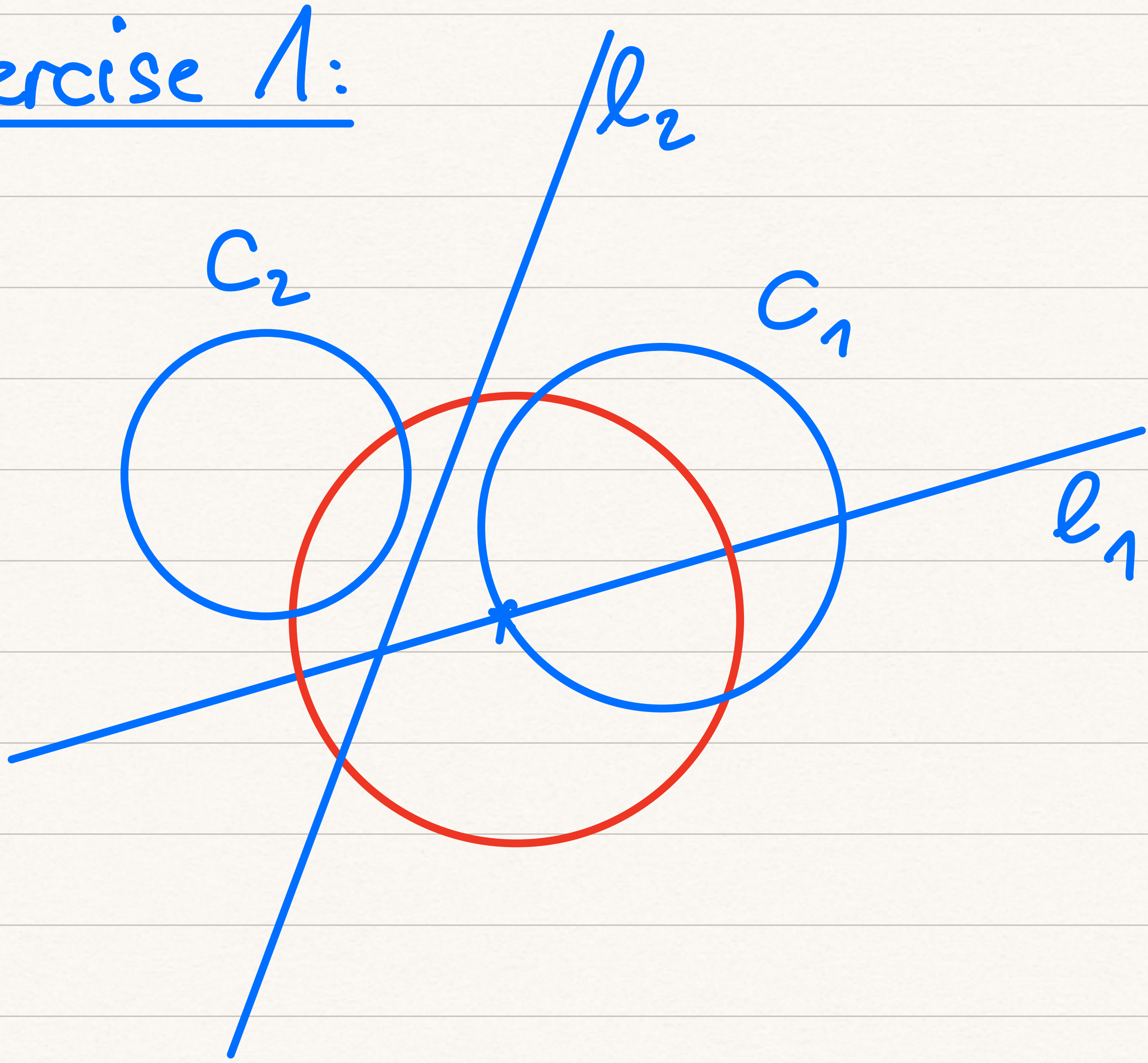
(3) By Thales, P' is on circle C' with diameter $A'B'$.

In summary,

$$P \in C \implies P' \in C'.$$

$$\boxed{\begin{array}{l}\text{Clines are sent to clines.}\\ \text{(Better: lines are just circles through } \infty.)\end{array}}$$

Exercise 1:



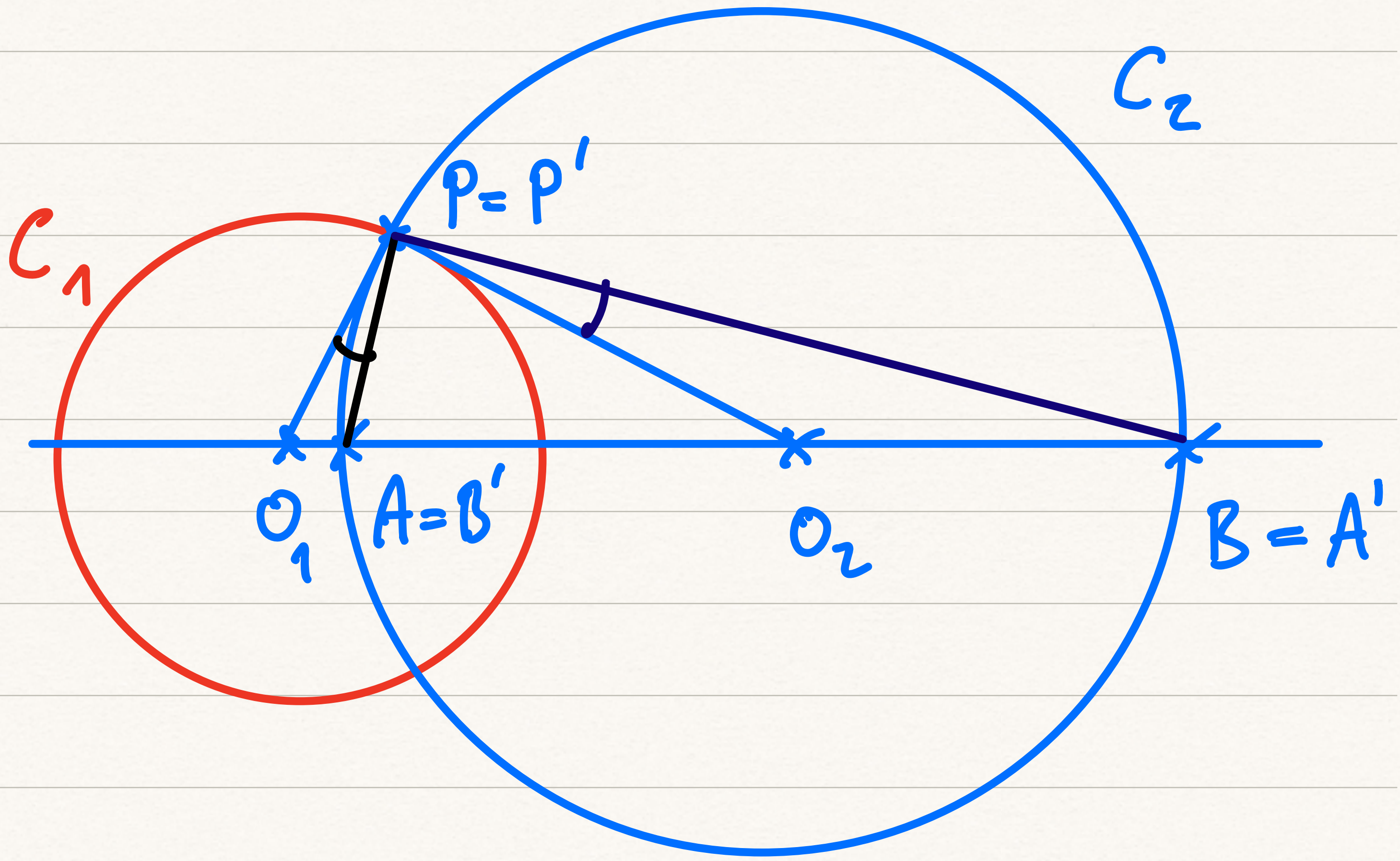
Sketch the images of C_1, C_2, ℓ_1, ℓ_2 under reflection!

Exercise 2:

C_1, C_2 circles with centers O_1, O_2

Let $P \mapsto P'$ inversion at C_1.

When is $C_2 = C_2'$?



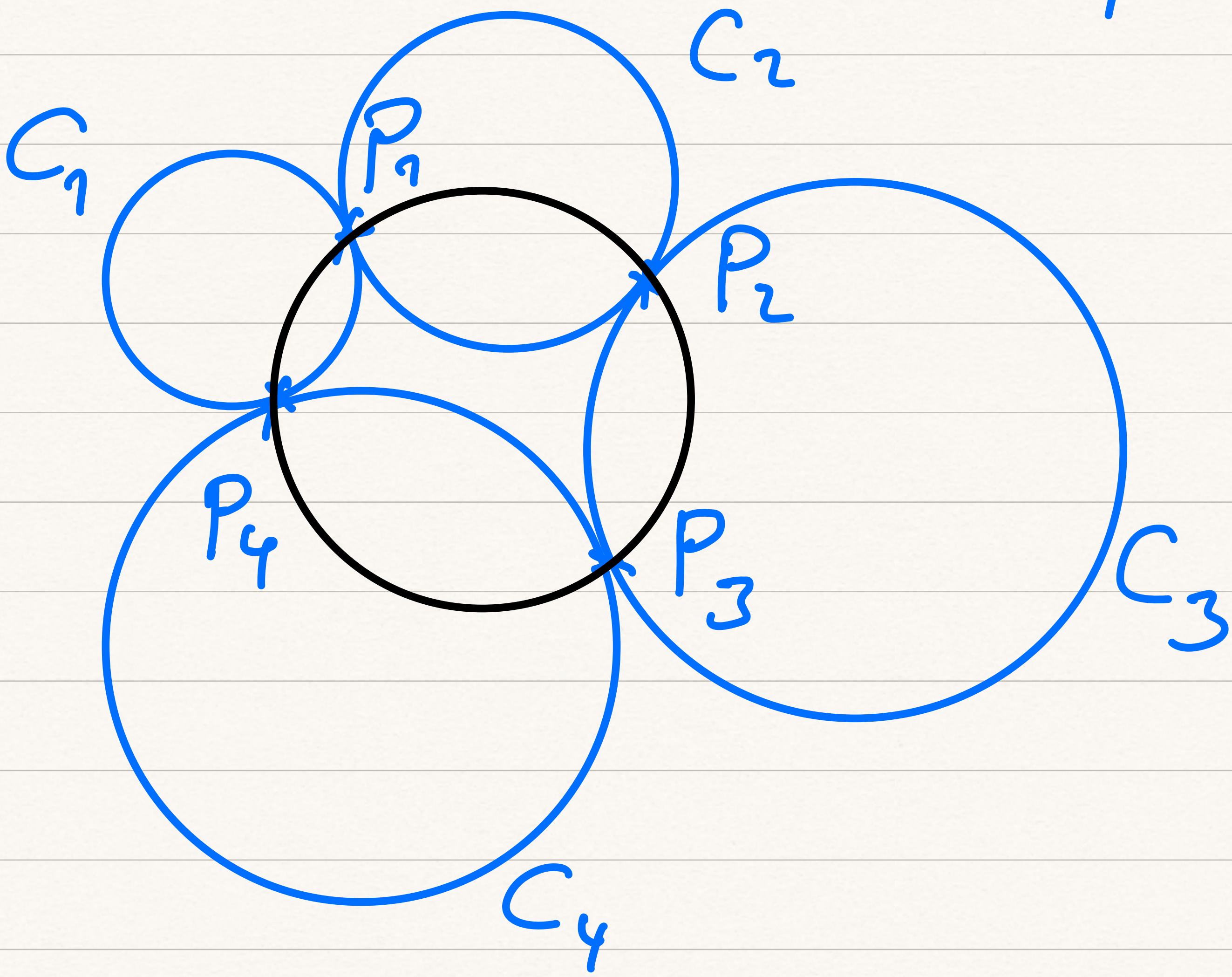
Hint: $A = B'$ and $B = A'$

$\Longrightarrow \measuredangle O_1 P A = \measuredangle O_2 P B$

Exercise 3:

Consider 4 non-intersecting circles C_1, C_2, C_3, C_4 such that

C_1 touches C_2 in P_1
$\vdots$
C_4 touches C_1 in P_4

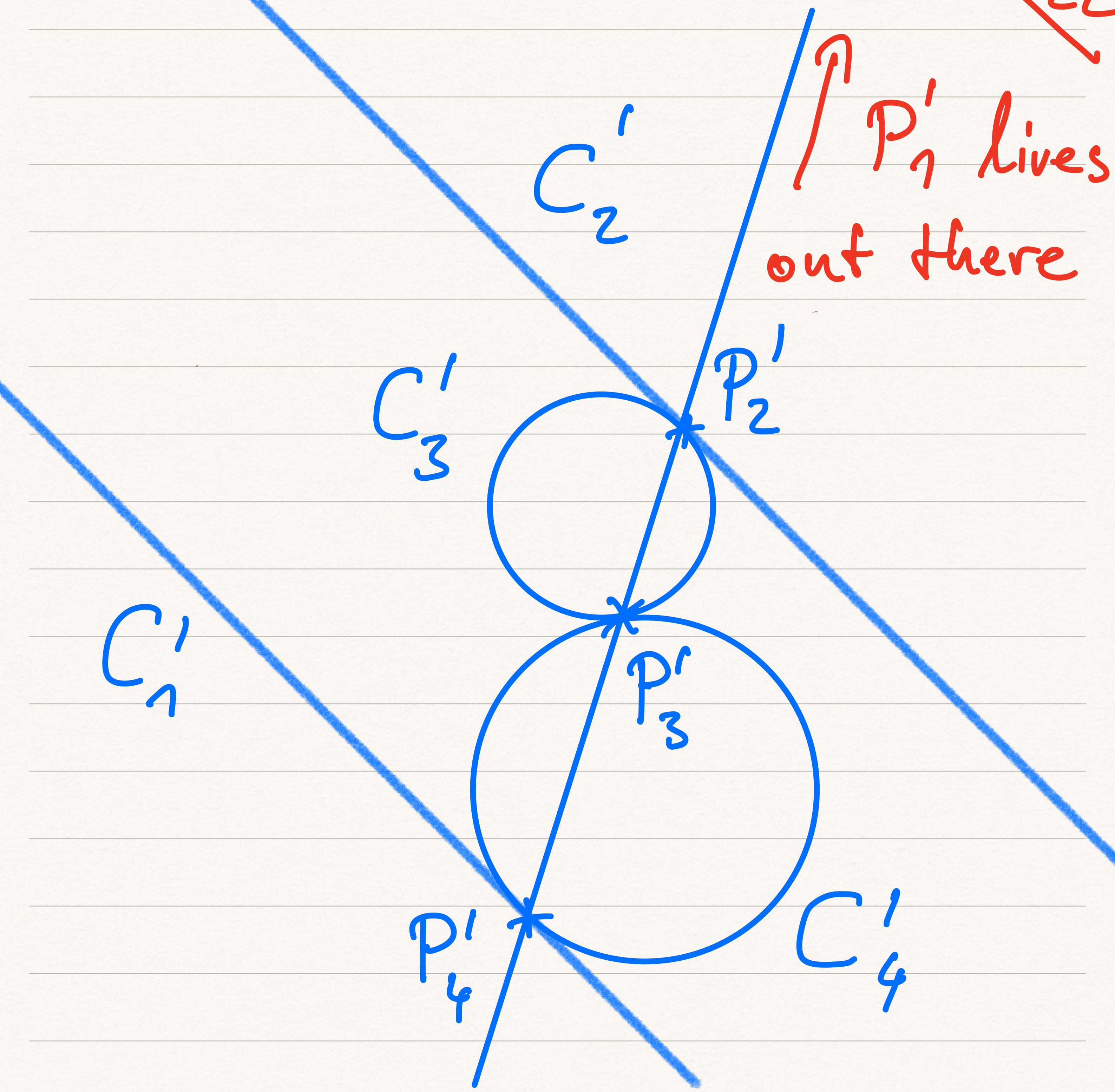


Prove that P_1, P_2, P_3, P_4 lie all on the same circle (using inversion).

First: Where would you invert?

— Usually where most circles meet, to turn these circles into lines.

So try any circle centered at say P_1.



P_1, P_2, P_3, P_4 on circle

$\Longleftrightarrow$ P_1', P_2', P_3', P_4' on line

Exercise 4: Pappus Circles in Shoemaker's Knife

images taken from →

fig. 1

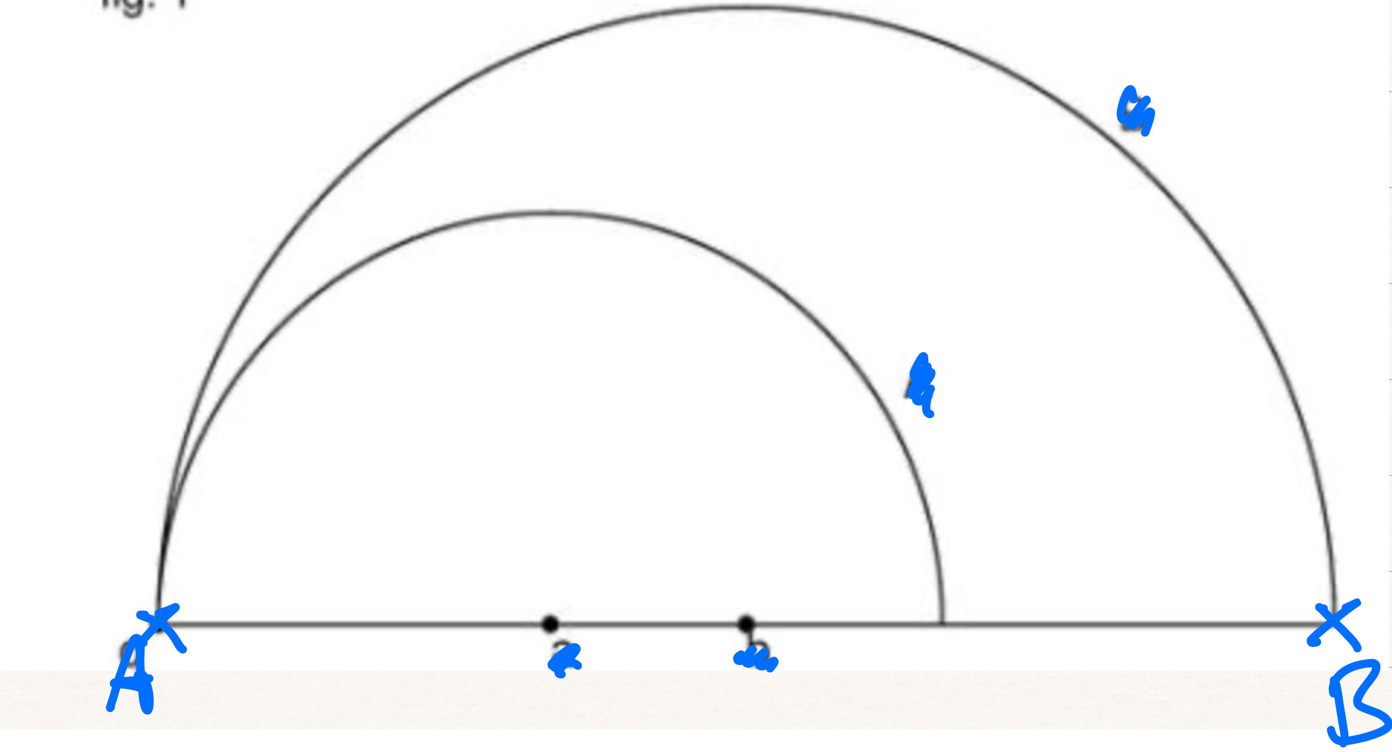
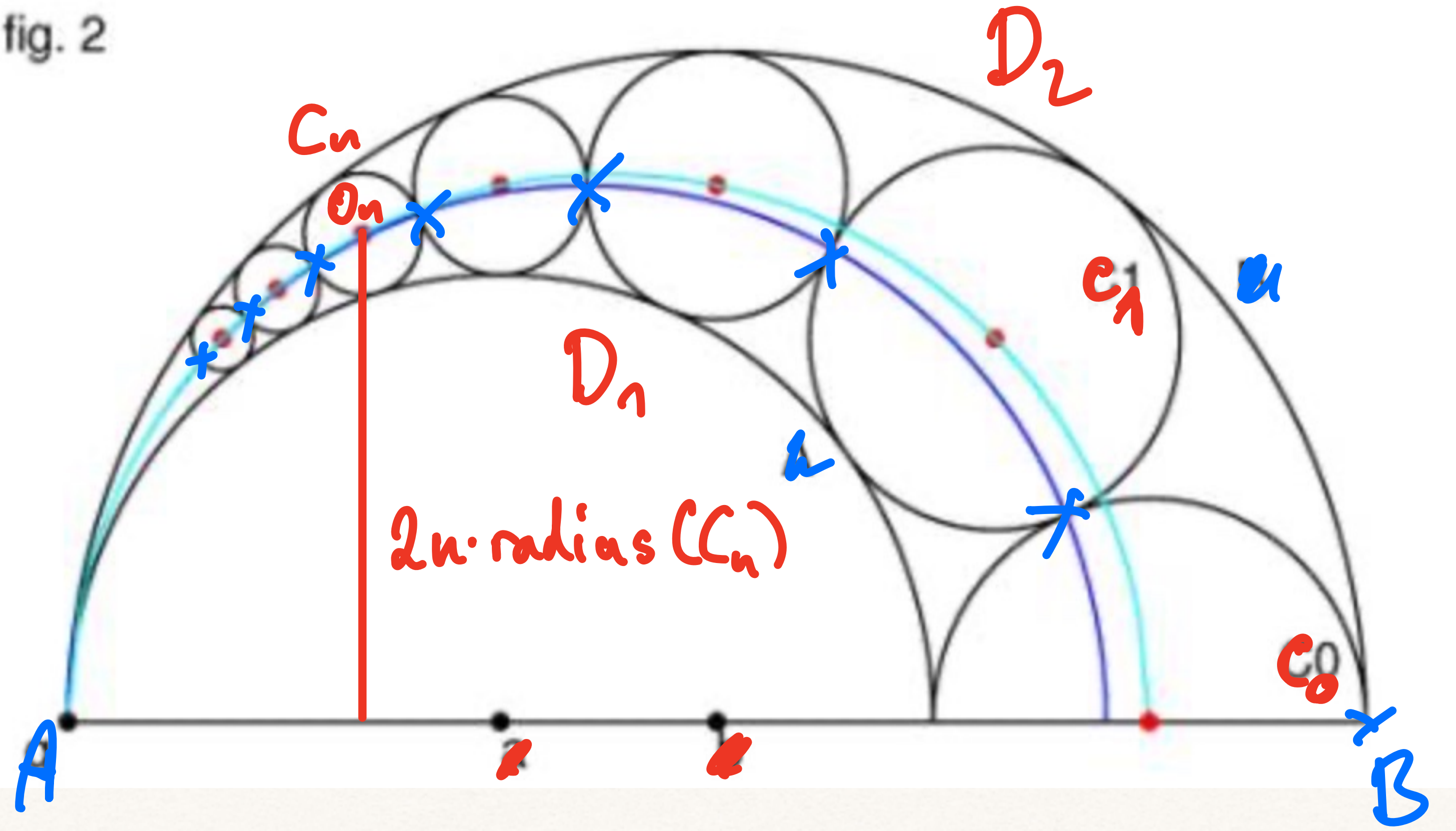


fig. 2



Prove that

(i) The points of tangency between the escribed circles all lie on a circle.

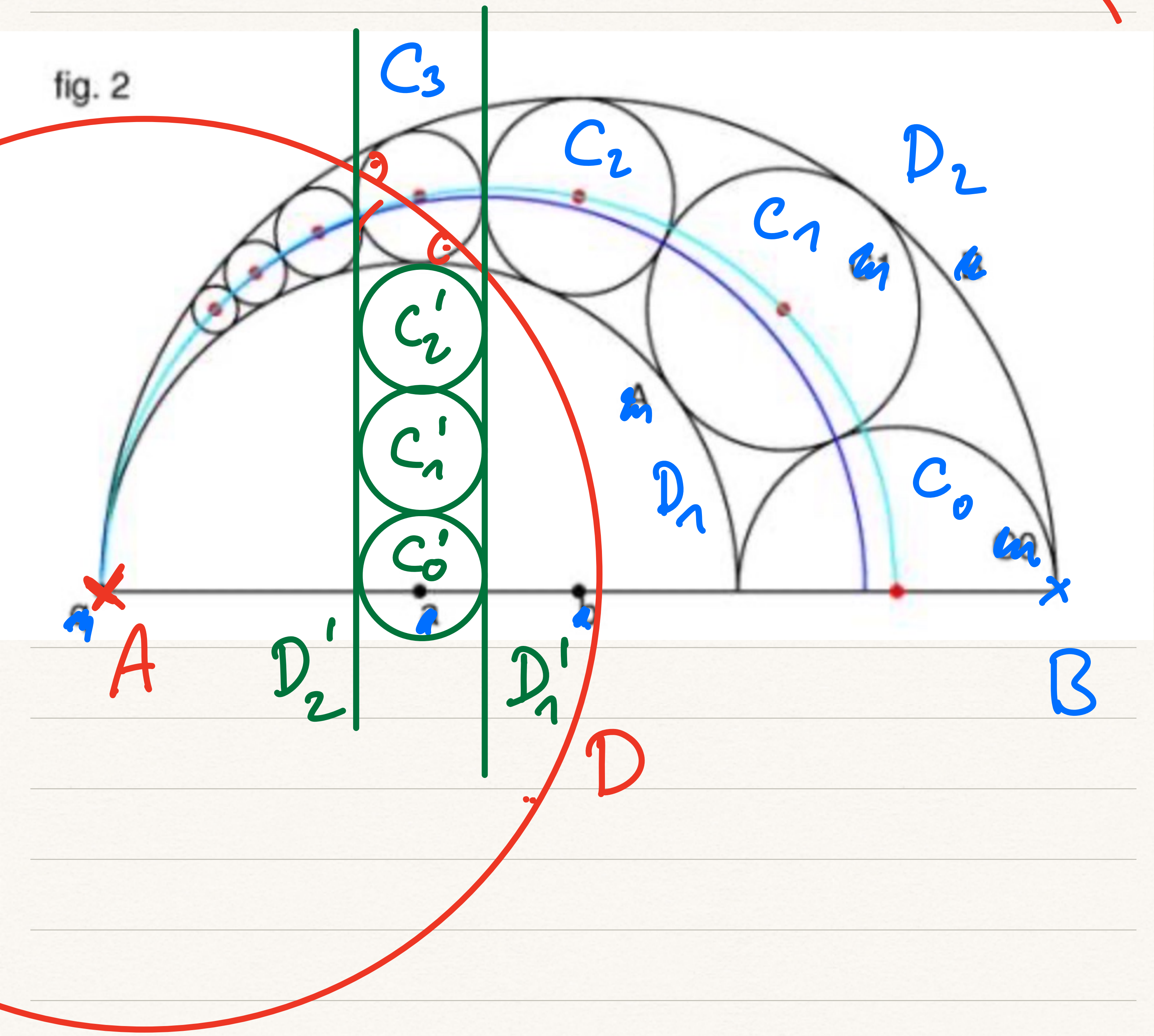
(ii) The distance between center of $\overline{AB}$ and O_n is $2 \cdot n \cdot \text{radius}(C_n)$.

Hint: Use inversion....

Hint 2: .... at A.

Hint 3: ... that keeps C_n fixed

fig. 2

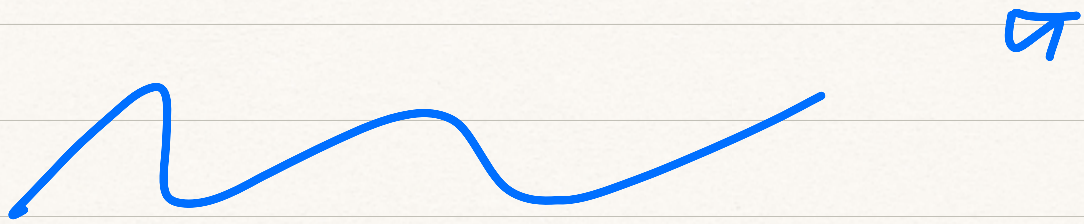


D = circle with center A orthogonal to C_n

Note conformality:

C_1: touches D_1, D_2

$\Rightarrow C_1'$: touches D_1', D_2'



References:

(i) Numberphile,
video on "Epic Circles"

(ii) Chen, Euclidean Geometry in Mathematical Olympiads, ch. 8
oriented towards math olympiads

(iii) Coexter, Geometry Revisited, ch. 5
a classic, but a little bit more analytical

(iv) many olympiad problems
by googling ....