THE GEOMETRY OF LIPSCHITZ-FREE SPACES

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Background

Let (M, d) be a *pointed* metric space, that is, a metric space with a fixed "base" point that we denote by 0. We define the *Lipschitz space* $\mathrm{Lip}_0(M)$ to be the vector space of all Lipschitz functions $f : M \to \mathbb{R}$ satisfying $f(0) = 0$. On this space the (optimal) Lipschitz constant of f defines a norm:

$$\|f\|_L = \mathrm{Lip}(f) = \sup\left\{\frac{f(x)-f(y)}{d(x,y)} \,:\, x, y \in M \text{ are distinct}\right\}, \quad f \in \mathrm{Lip}_0(M),$$

and with respect to this norm $\mathrm{Lip}_0(M)$ becomes a complete normed vector space (i.e. a *Banach space*) over $\mathbb{R}$.

Let the vector space

$$\mathrm{Lip}_0(M)^* = \{\psi : \mathrm{Lip}_0(M) \to \mathbb{R} \,:\, \psi \text{ is linear and continuous}\},$$

equipped with the norm

$$\|\psi\| = \sup\{|\psi(f)| \,:\, \|f\|_L \leqslant 1\},$$

denote the *dual space* of continuous linear functionals on $\mathrm{Lip}_0(M)$. There is a natural embedding $\delta : M \to \mathrm{Lip}_0(M)^*$ that is given by evaluation: $\delta(x)(f) = f(x)$, $f \in \mathrm{Lip}_0(M)$. It can be shown that δ is a (non-surjective) *isometry*, that is

$$\|\delta(x) - \delta(y)\| = d(x, y), \quad x, y \in M.$$

We define the *Lipschitz-free space* (or simply *free space*) over M to be the *closed linear span* of the image of M under δ:

$$\mathcal{F}(M) = \overline{\mathrm{span}(\delta(M))} \subset \mathrm{Lip}_0(M)^*.$$

In other words, we take the span of the set $\delta(M)$ to get a vector subspace of $\mathrm{Lip}_0(M)^*$, and then take the closure of that (with respect to the metric induced by the norm), which will give us a closed vector subspace of $\mathrm{Lip}_0(M)^*$. Being a closed vector subspace of a Banach space, $\mathcal{F}(M)$ is again a Banach space. It turns out that $\mathcal{F}(M)$ is an *isometric predual* of $\mathrm{Lip}_0(M)$, that is, there exists a linear surjective isometry $J : \mathrm{Lip}_0(M) \to \mathcal{F}(M)^*$.

The elementary molecules turn out to be particularly important free space elements. We set

$$\widetilde{M} = \{(x, y) \in M \times M \,:\, x \neq y\},$$

and given $(x, y) \in \widetilde{M}$, we define the *(normalised) elementary molecule*

$$m_{xy} = \frac{\delta(x) - \delta(y)}{d(x, y)}.$$

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Since δ is an isometric embedding, each elementary molecule belongs to the *unit sphere*

$$S_{\mathcal{F}(M)} = \{m \in \mathcal{F}(M) \,:\, \|m\| = 1\}$$

of $\mathcal{F}(M)$, that is, the set of all points $m \in \mathcal{F}(M)$ having distance 1 from the origin.

A wealth of information about Lipschitz and Lipschitz-free spaces can be found in the authoritative text [3]. Free spaces possess a "universal property" analogous to that of a free group, hence their name. They lie on the interface between functional analysis, optimal transport theory and metric geometry. Free spaces (and their duals) are arguably the canonical way to express metric spaces in functional analytic terms, analogously to how compact Hausdorff spaces and measure spaces can be expressed using $C(K)$-spaces and L_p-spaces, respectively.

While free spaces have been known since [1], they have been studied intensively by members of the functional analysis community ever since the publication of the seminal papers of Godefroy and Kalton (e.g. [2]), and remain a very active research area. One of the reasons for this is that, despite the relative ease with which free spaces can be defined, their structure is complicated and elusive, and many pertinent and "elementary" questions concerning their theory remain unanswered.

The geometry of two-dimensional subspaces of Lipschitz-free spaces

As stated above, there are still many unanswered "elementary" questions about the geometry and structure of free spaces. For example, we lack an explicit or "closed-form" expression for the free space norm, even when restricted to two-dimensional subspaces. The aim of this project is to try to solve the following problem.

Problem Let $(x, y), (u, v) \in \widetilde{M}$ be distinct, and consider the (at most) 2-dimensional subspace

$$X = \operatorname{span}\{m_{xy}, m_{uv}\} = \{am_{xy} + bm_{uv} \,:\, a, b \in \mathbb{R}\}$$

of $\mathcal{F}(M)$. Find an explicit expression for the norm $\|am_{xy} + bm_{uv}\|$ of $am_{xy} + bm_{uv}$ in terms of $(x, y), (u, v) \in \widetilde{M}$ and $a, b \in \mathbb{R}$.

This problem has essentially been solved in the case $a = b = 1$; see [4, Lemma 1.2]. The natural initial approach to the problem above would be to study [4, Lemma 1.2] and seek generalisations thereof. Another approach might be to restrict attention to certain metric spaces whose metric is easier to work with. Time allowing, the problem could be naturally extended to subspaces of $\mathcal{F}(M)$ spanned by finitely many elementary molecules. It is possible that the results could lead to the publication of a paper.

References

[1] R. F. Arens and J. Eells, *On embedding uniform and topological spaces*, Pac. J. Math. **6** (1956), 397–403. Paper

[2] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. **159** (2003), 121–141. Paper

[3] N. Weaver, *Lipschitz algebras*, 2nd ed., World Scientific Publishing Co., River Edge, NJ, 2018. DOI

[4] T. Veeorg, *Characterizations of Daugavet points and delta-points in Lipschitz-free spaces*, Studia Math. **268** (2023), 213–233. Paper