EXPERIMENT 7

Sinusoidal Response of the LCR Resonant Circuit

Introduction

The vast majority of households and power distribution systems operate with Alternating Current (ac), a consequence of the fact that ac voltages can be controlled with transformers, unlike dc. High voltage ac can therefore be transmitted efficiently across power lines into homes, where it is then stepped down to a safer level for domestic use. Any appliance that is plugged into the wall uses ac. Radio and television circuits also make extensive use of ac.

AC circuits deal with time-varying voltage and current signals and the basic circuit elements are capacitors (C) and inductors (L). Resistors (R) act as dissipative elements in the circuit. Each element has its own impedance to the flow of alternating current and when the total impedance is minimised the circuit is said to be in resonance. The resonant behaviour of LCR circuits has many important applications in modern communications and is analogous to resonance phenomena observed in other branches of physics, such as in mechanical systems and in atomic and nuclear physics. In this experiment the following aspects of the series LCR circuit are studied:

1. The gain and phase, quantities which characterise the circuit, are measured and compared with the theoretical values

2. The factors affecting resonance and the behaviour of the circuit in resonance are established

3. The application of the LCR circuit to radio tuning is established by measuring the circuit response curve for a number of resistance values

Theory

When inductors and capacitors are used in a circuit there may be a phase shift between the current in the circuit and the driving ac voltage, but the magnitude of the current is still proportional to the magnitude of the voltage. In ac circuits, the constant of proportionality between current and voltage is no longer simply R, the resistance, but a sum of terms due to contributions to the total

(a) Time domain (b) Frequency domain

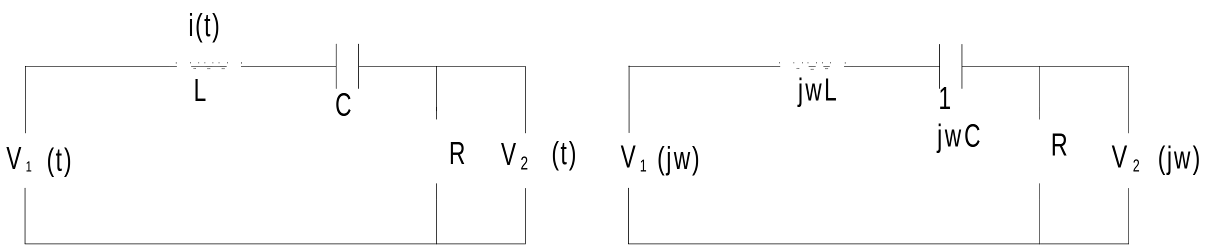

Figure 7.1: *The series LCR circuit*

impedance (Z) from the capacitor C and the inductor L, as well as R. The impedance of an inductor, X_L, is given by $j\omega L$ where $j = \sqrt{-1}$ and ω is the driving frequency. The impedance of a capacitor, X_C, is given by $\frac{1}{j\omega C}$. The total impedance is then:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The current amplitude is given by $I = V/Z$, the ac equivalent of Ohm's Law. When $X_L = X_C$, Z is minimised and the circuit is in resonance.

The time domain and frequency domain representations of the series LCR circuit, together with the appropriate circuit equations, are set out below.

$$v_1 = L\frac{di}{dt} + \frac{1}{C}\int i\,dt + Ri \qquad V_1 = j\omega LI + \frac{I}{j\omega C} + RI$$

$$\frac{d^2v_2}{dt^2} + \frac{R}{L}\frac{dv_2}{dt} + \frac{1}{LC}v_2 = \frac{dv_1}{dt} \qquad V_2 = IR = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}V_1$$

We see that the use of (complex) phasors in the frequency domain (b) eliminates the complicated integro-differential equations of the time domain (a). The sinusoidal response function $H(j\omega)$ is defined by the ratio of the output to input phasor in the frequency domain:

$$H(j\omega) \equiv \frac{V_2}{V_1} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega CR} \tag{7.1}$$

and theoretical analysis shows that, if

$$v_1(t) = A\cos(\omega t) \text{ and } v_2(t) = B\cos(\omega t + \varphi)$$

then

$$\text{Gain} \equiv \frac{B}{A} = |H| = \omega CR\left[(1 - \omega^2 LC)^2 + (\omega CR)^2\right]^{-\frac{1}{2}} \tag{7.2}$$

and

$$\text{Phase} \equiv \varphi = \angle H = 90^\circ - \tan^{-1}\left[\frac{\omega CR}{(1 - \omega^2 LC)}\right] \tag{7.3}$$

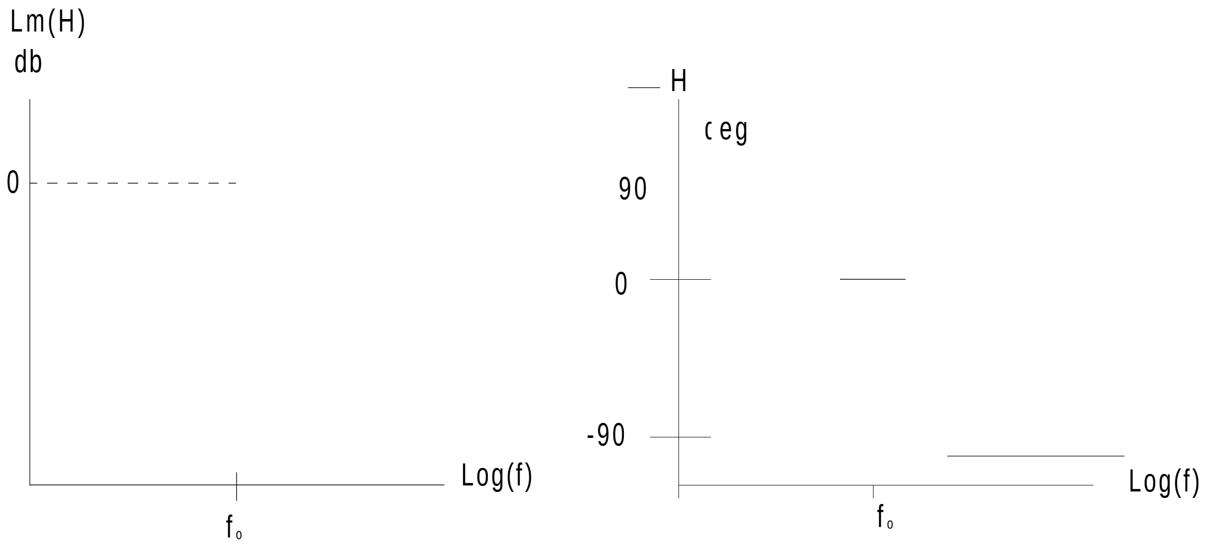

Figure 7.2: *Typical Bode plots for the series LCR circuit*

Thus the magnitude and angle of H(jω) at any frequency determine the voltage gain and phase shift for a sinusoid passing through the network at that frequency. So H(jω) entirely defines the network response for all frequencies, and is a fundamental circuit description for linear circuits.

A plot of magnitude (usually as Lm(H) = 20 $\log_{10}(|H|)$ in db) and angle of H as functions of log(frequency) is called a Bode plot for the system, and in this experiment the Bode plot is determined experimentally for the series LCR circuit and compared with theoretical predictions.

For the LCR case, the nature of the response curves (gain and phase) depend very much on the relative values of L, C, and R. Firstly, from Eq. 7.2, we see that the gain reaches a maximum of unity (or 0 db) when $1 - \omega^2 LC = 0$. This defines the resonant frequency as

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{7.4}$$

Secondly, we see from Eq. 7.3 that the phase shift between the input and output signals varies from a 90^0 lead at low frequency ($\ll f_0$) to a phase lag of 90^0 at high frequency ($\gg f_0$), passing through the 0^0 value exactly at f_0. Typical plots are illustrated in Fig. 7.2.

The "sharpness" of the resonance is related to the narrowness of the Lm (gain) curve, and to the rapidity of the phase transition through the resonant frequency. A quantitative measure of this "sharpness" is provided by the quality factor, or Q-factor, defined by:

$$Q \equiv \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} \tag{7.5}$$

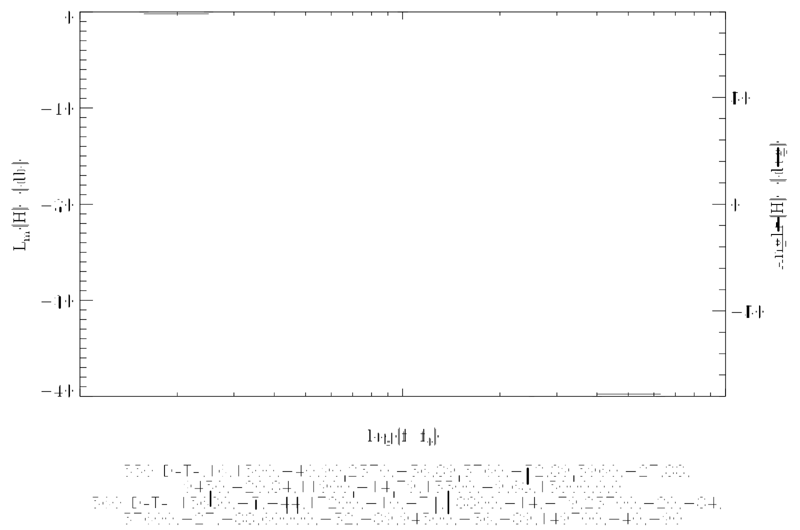

Figure 7.3: *An example of the computer printout for the given values of L, C, R and Q. The log of the ratio of the frequency to the resonant frequency is plotted against the phase angle (_H) and the magnitude (Lm). Also listed is a printout of the input data from lines 550 and 560. Note there is no comma at the end of each data statement.*

For high Q (say > 10), the circuit shows high selectivity for frequencies at or very close to the resonant frequency, and is often referred to as a tuned circuit. In this form, it provides the basis for tuning a radio set to different transmission frequencies, for example.

Experimental Procedure

Theoretical predictions for the LCR circuit

On the desktop double-click on “LCR Plot”. In the lower left of the window you can enter values for L, C and R; take L = 1 mH and C = 0.1 μF and vary R to see the effects on the resonance plots and Q factor. The program will calculate the resonance frequency and Q and will plot graphs of the frequency response and phase angle. The graph can be printed and saved in a number of formats, including .jpg and .bmp. Note the effect of different values or R on the Q values for a given capacitance.

Set up the apparatus as shown in Fig. 7.4, with C set to 0.1μF and R to 10Ω. Using the sinewave output of the oscillator, find the resonant frequency, f_0, by observing v_2 while the amplitude of v_1 is kept constant and its frequency varied. You should notice that the phase shift is zero at resonance. Use this value to calculate the value of L, from Eq. 7.4. Make an alternative (more accurate) measurement of f_0 using the oscilloscope rather than the frequency generator dial. Estimate the error on your measurement of L.

Measurements for the Bode plot

In this part of the experiment, the behaviour of the LCR circuit is studied as the input frequency is varied. The values of L, C and R are kept fixed.

Choosing logarithmic increments in frequency over 2 decades of frequency around f_0, (i.e. from $0.1f_0$ to $10f_0$), measure at each frequency:

1. The input and output peak to peak voltage
2. The time difference ΔT between peaks in the input and output

Ensure you take substantially more readings close to resonance. Ideally, you should choose a set of frequencies that will give you data points symmetrically placed about the resonant frequency on a log axis scale.

The two quantities that characterise the circuit, the gain (Lm(H)) and the phase (_H) can be calculated from the measured quantities v_1, v_2 and ΔT from:

$$\text{Lm(H) (dB)} = 20\, log_{10} \frac{v_2}{v_1}$$

$$\text{\_H } (^{\circ}) = 360 \times \text{f} \times \Delta T$$

Make a table of your data as shown:

f	: a, b, c, ...Hz
Lm(H)	: p, q, r, ...db
_ H	: x, y, z, ...deg

Choose values $\pm$ 1 decade of the resonant frequency. When you are ready to plot your data, click "Enter Data" button in the "LCR Theoretical" window to open a new window. Enter your calculated data points in the boxes on the lower left and click "Add" after each set of three data points; the date you have entered will be added to the list boxes in the lower centre of the window. Enter your data carefully – it cannot be edited after you click "Add". If you make a mistake the "Clear" button will clear all entered data. When you have finished entering data click "Plot Graph". Note that the frequency scale is logarithmic. The graph can be saved or printed as before.

You can now insert the known values of L and C and vary the value of R to see if you can achieve a reasonable agreement between the theoretical plot and your experimental points. Comment on how good or bad the fit is, and on the value of R required for the best fit. Is this R = 10Ω, and if not, why not?

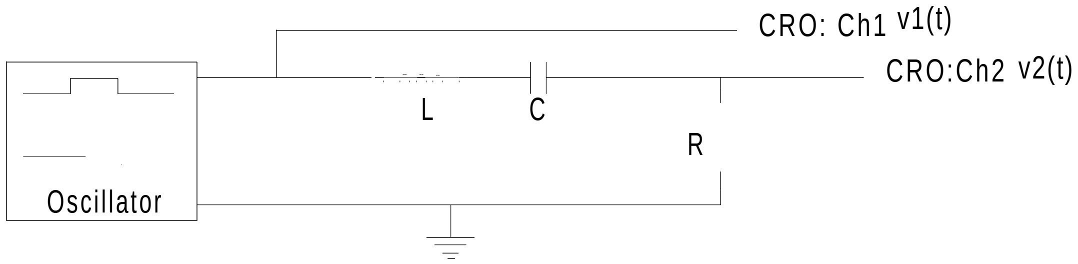

Figure 7.4: *Experimental Layout*

Measurement of the circuit response curves

In this section we use the circuit like a radio to tune in on a specific input frequency by adjusting the resonant frequency of the circuit. Recall that the LCR resonant frequency is given by $f_{\mathrm{LCR}} = \frac{1}{2\pi\sqrt{L_0 C}}$ where L_0 is the (fixed) inductance of the inductor, as measured earlier. So by varying C we can vary the resonant frequency: $f_{\mathrm{LCR}} \propto \frac{1}{\sqrt{C}}$. We have previously observed that maximum gain occurs when the input frequency matches the resonant frequency of the circuit. Conversely the input signal at fixed frequency will be amplified to the greatest extent when the LCR circuit is tuned (by varying the capacitance) to match this input frequency.

Method

1. Set R = 10 Ω and the input frequency to the f_0 measured above.
2. Vary C over a range of values around resonance. Use as fine a scale as possible.
3. At each value of f_{LCR} record the output voltage amplitude from the oscilloscope.
4. Set R = 30 Ω, keep the input frequency fixed and repeat steps 2 and 3 above.

Plotting the current as a function of ω gives the circuit response curve. A set of theoretical response curves for this system is shown in Fig. 7.5. Plot your measured response curves on graph paper (or the computer) and calculate the Q value for each value of resistance. If time permits, generate a set of theoretical curves such as those in Fig. 7.5 using Mathcad and compare them with your experimentally determined values.

How a radio works:
The shape of the response curve is important in the design of radio and television receiving circuits. The sharply peaked curve is what makes it possible to discriminate between two stations

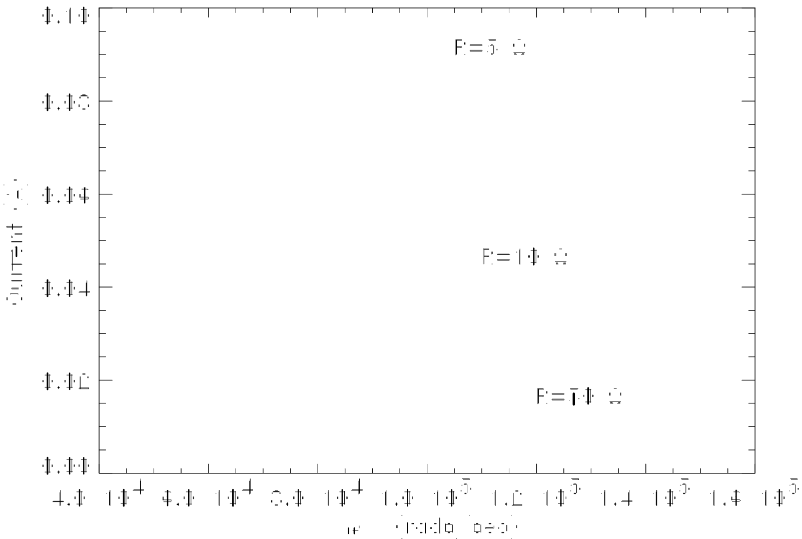

Figure 7.5: *Theoretical response curves for the LCR series circuit assuming an input voltage amplitude of 0.45 V*

broadcasting on adjacent frequency bands. However, if the peak is too sharp, then some of the information is lost, such as the high frequencies in music. The shape of the resonance curve is also related to overdamped and underdamped oscillations which are studied in the experiment on ‘Signal Processing with RC and LCR Circuits’. By varying the capacitance or inductance in a radio circuit you tune the circuit so the response is a maximum. In this sense the LCR circuit acts as a filter. Expensive tuners use low resistance components to obtain the sharpest response.

References

1. Electricity and Magnetism, W.J. Duffin

2. Electricity and Magnetism (Berkeley Physics Series), Purcell University Physics, Young and Freedman